CALCULUS: THE EXERCISES

MATH 150: CALCULUS WITH ANALYTIC GEOMETRY I

VERSION 1.3

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- You may download these and other course notes, exercises, and exams.

Feel free to send emails with suggestions, improvements, tricks, etc.

LICENSING

PARTIAL BIBLIOGRAPHY / SOURCES

Calculus: Anton, Edwards/Penney, Larson, Stewart, Swokowski, Thomas

People: Ken Kuniyuki, Laleh Howard, Tom Teegarden, and many more.

"END OF CHAPTER" MARK

Page E.1.3 is the last page of the Exercises for Chapter 1. Therefore, in the upper right hand corner, there is an additional period: "E.1.3."

This is to help people figure out if they have all the pages for a particular chapter.

CHAPTER 1: REVIEW

FUNCTIONS

1) Assuming $f(x) = x^2$, evaluate $f(-4)$, and evaluate (and expand) $f(a+h)$.

2) If $f(t) = 4t^5 - 13t + 4$, is f a polynomial function? Rational? Algebraic?

3) If $g(w) = \dfrac{w-1}{w^3 + 2w - 3}$, is g a polynomial function? Rational? Algebraic?

4) If $h(x) = \sqrt[3]{x} - x^{2/5}$, is h a polynomial function? Rational? Algebraic?

5) If $f(x) = \sin x$, is f a polynomial function? Rational? Algebraic?

DOMAIN AND RANGE

6) For each function rule below, write the domain of the corresponding function in interval form.

   a) $f(x) = 7x^9 + 4x^6 - 12$

   b) $g(t) = t^{2/3} + 4$

   c) $h(w) = \dfrac{\sqrt[3]{w-4}}{2w^2 - 3w - 2}$

   d) $f(t) = \dfrac{\sqrt{t+2}}{t-5}$

   e) $g(t) = \dfrac{t-5}{\sqrt{t+2}}$

   f) $h(x) = \sqrt{7-x}$

   g) $f(r) = \sqrt{r^2 - 2r - 3}$

   Review Section 2.7 on Nonlinear Inequalities in the Precalculus notes. See the domain discussion coming up in Section 2.2, Example 5.

7) Consider $f(x) = \sqrt{x} - 2$. Graph $y = f(x)$. What is the domain of f in interval form? What is the range of f in interval form?

SYMMETRY

Topic 3, Part A can help with the trigonometry.

8) If $f(x) = x^4 - 3x^2 + 4\cos x$, then the graph of $y = f(x)$ in the usual *xy*-plane is symmetric about what? Why? (What kind of function is f?)

9) If $f(x) = 2x^7 - x - 4\sin x$, then the graph of $y = f(x)$ in the usual *xy*-plane is symmetric about what? Why? (What kind of function is f?)

10) If $g(t) = t^{2/3} + 4$, then is the function g even, odd, or neither?

11) If $h(r) = r\tan r$, then is the function h even, odd, or neither?

12) If $f(x) = x^3 - 2x + 1$, then is the function f even, odd, or neither?

COMPOSITIONS OF FUNCTIONS

13) Let $f(u) = u^2 + \frac{1}{u+1}$ and $g(x) = x^4 + x^2$. Find $(f \circ g)(x)$ and $\mathrm{Dom}(f \circ g)$. You do <u>not</u> have to express $(f \circ g)(x)$ as a single fraction.

14) Find rules for functions f and g so that $(f \circ g)(x) = f(g(x)) = (x^4 + x)^8$. (Do <u>not</u> let f or g be the identity function.)

15) Find rules for functions f and g so that $(f \circ g)(t) = f(g(t)) = \sqrt[4]{\frac{1}{t}}$. (Do <u>not</u> let f or g be the identity function.)

16) Find rules for functions f and g so that $(f \circ g)(r) = f(g(r)) = \sin(r^2)$. (Do <u>not</u> let f or g be the identity function.)

TRIGONOMETRY

17) Evaluate the following; write "undefined" when appropriate.

a) $\cot \pi$

b) $\sec \pi$

c) $\csc\left(\dfrac{3\pi}{4}\right)$

d) $\sec\left(\dfrac{7\pi}{6}\right)$

e) $\sin\left(\dfrac{5\pi}{3}\right)$

f) $\tan\left(\dfrac{5\pi}{3}\right)$

18) Verify the following trigonometric identities.

a) $\dfrac{\sin(2x)}{\cos^2 x} = 2(\sin x)(\sec x)$

b) $\dfrac{1}{\sec^2 x - 1} = \cot^2 x$

19) Solve the following trigonometric equations; find all real solutions, and write the solution set in set-builder form.

a) $2\sin^2 x + 3\sin x = -1$

b) $2\cos(3x) - 1 = 0$

KNOW THE FOLLOWING

- Domains, ranges, and graphs of the six basic trigonometric functions.

- Fundamental and Advanced Trigonometric Identities in Ch.1, except you do not have to memorize the Product-To-Sum Identities, nor the Sum-To-Product Identities.

CHAPTER 2: LIMITS AND CONTINUITY

When asked to give a limit, give a real number or ∞ or $-\infty$ when appropriate.
If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE."

SECTION 2.1: AN INTRODUCTION TO LIMITS

Assume that a is a real constant.

BASIC LIMIT THEOREM FOR RATIONAL FUNCTIONS

1) Evaluate $\lim_{x \to -2} \left(3x^4 - x^3 + 1\right)$.

2) Evaluate $\lim_{r \to 3} \dfrac{2r + 5}{r^2 - 2}$.

3) Evaluate $\lim_{x \to \frac{1}{3}} 10$.

4) Evaluate $\lim_{t \to 11} \pi^2$.

ONE- AND TWO-SIDED LIMITS; EXISTENCE OF LIMITS; IGNORING THE FUNCTION AT *a*

5) We have discussed how the numerical / tabular method can help us guess at limits. We will see here how this method can be misleading at times!
   Let $f(x) = x - 0.0001$.

   a) Evaluate $f(1)$, $f(0.1)$, and $f(0.01)$.

   b) Evaluate $\lim_{x \to 0} f(x)$. Is the result obvious from the function values in a)?

6) Evaluate: a) $\lim_{r \to 3^+} \dfrac{2r + 5}{r^2 - 2}$, and b) $\lim_{r \to 3^-} \dfrac{2r + 5}{r^2 - 2}$. Compare with Exercise 2.

7) Yes or No: If the one-sided limit $\lim_{x \to a^+} f(x)$ exists, then must the two-sided limit $\lim_{x \to a} f(x)$ exist?

- If your answer is "Yes," then explain why.

- If your answer is "No," then give a counterexample. A counterexample is a situation where the hypothesis (assumption) holds true, but the conclusion does not hold true. Here, the hypothesis is "the one-sided limit $\lim_{x \to a^+} f(x)$ exists," and the conclusion is "the two-sided limit $\lim_{x \to a} f(x)$ must exist."

- An if-then statement is true if and only if no such counterexamples exist.

8) Yes or No: If the two-sided limit $\lim_{x \to a} f(x)$ exists, then must the one-sided limit $\lim_{x \to a^+} f(x)$ exist?

9) Yes or No: If $f(a)$ exists, then must $\lim_{x \to a} f(x)$ equal $f(a)$?

- If your answer is "Yes," then explain why.
- If your answer is "No," then give a counterexample.

10) Assume $f(a)$ does not exist.

- If $\lim_{x \to a} f(x)$ cannot exist, write "cannot exist" and explain why it cannot exist.

- If $\lim_{x \to a} f(x)$ might exist, write "might exist" and give an example.

11) Let the function g be defined piecewise as follows: $g(x) = \begin{cases} x+2, & \text{if } x < -1 \\ x^2 - 1, & \text{if } -1 \le x < 2 \\ \sqrt{x+1}, & \text{if } x \ge 2 \end{cases}$

a) Draw the graph of $y = g(x)$. Remember transformations from Section 1.4 of the Precalculus notes!

b) Evaluate: $\lim_{x \to -1^-} g(x)$, $\lim_{x \to -1^+} g(x)$, and $\lim_{x \to -1} g(x)$.

c) Evaluate: $\lim_{x \to 2^-} g(x)$, $\lim_{x \to 2^+} g(x)$, $\lim_{x \to 2} g(x)$, and $\lim_{x \to 3} g(x)$.

12) Let $f(x) = \dfrac{|x-3|}{x-3}$.

a) Draw the graph of $y = f(x)$. Remember transformations from Section 1.4 of the Precalculus notes!

b) Evaluate: $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$, and $\lim_{x \to 3} f(x)$.

13) (Charles's Law for Ideal Gases). Assuming that we have an ideal gas occupying volume V_0 (measured in liters, let's say) when the temperature of the gas is 0° Celsius, and assuming that the gas is under constant pressure, the volume of the gas when its temperature is T degrees Celsius is given by:

V, or $V(T) = V_0\left(1 + \dfrac{T}{273.15}\right)$. Absolute zero is -273.15° Celsius.

a) Evaluate $\lim_{T \to -273.15^+} V(T)$, and interpret the result.

b) Discuss $\lim_{T \to -273.15^-} V(T)$.

SECTION 2.2: PROPERTIES OF LIMITS and ALGEBRAIC FUNCTIONS

1) Evaluate $\lim_{x\to 3}\left(\dfrac{2x^2\sqrt{x}-3x}{(x+1)^2}+x\right)$.

2) Evaluate: a) $\lim_{x\to 0^+}\sqrt{-x}$, b) $\lim_{x\to 0^-}\sqrt{-x}$, and c) $\lim_{x\to 0}\sqrt{-x}$.

3) Evaluate: a) $\lim_{x\to 0^+} x^{2/3}$, b) $\lim_{x\to 0^-} x^{2/3}$, and c) $\lim_{x\to 0} x^{2/3}$.

4) Evaluate:

a) $\lim_{x\to 4^+}\sqrt{x-4}$

b) $\lim_{x\to 4^-}\sqrt{x-4}$

c) $\lim_{x\to 4}\sqrt{x-4}$

d) $\lim_{x\to 4^-}\sqrt{4-x}$

e) $\lim_{x\to 9}\sqrt{x-4}$

f) $\lim_{x\to 3}\sqrt{x-4}$

g) $\lim_{x\to -2}\sqrt[3]{x+2}$

5) Evaluate:

a) $\lim_{t\to -4^+}\sqrt{t^2+3t-4}$

b) $\lim_{t\to -4^-}\sqrt{t^2+3t-4}$

c) $\lim_{t\to -4}\sqrt{t^2+3t-4}$

d) $\lim_{t\to 1^+}\sqrt{t^2+3t-4}$

e) $\lim_{t\to 1^-}\sqrt{t^2+3t-4}$

f) $\lim_{t\to 1}\sqrt{t^2+3t-4}$

g) $\lim_{t\to 2^+}\sqrt{t^2+3t-4}$

6) Yes or No: If $\lim_{x\to 2} f(x)=10$, then must $\lim_{x\to 2}\left[5f(x)-4\right]=46$?

- If your answer is "Yes," then explain why.
- If your answer is "No," then give a counterexample.

7) Assume $\lim_{x\to a^+}\sqrt{f(x)}=0$.

- If $\lim_{x\to a^-}\sqrt{f(x)}$ cannot exist, write "cannot exist" and explain why it cannot.
- If $\lim_{x\to a^-}\sqrt{f(x)}$ might exist, write "might exist" and give an example.

SECTION 2.3: LIMITS AND INFINITY I

HORIZONTAL ASYMPTOTES (HAs) and "LONG-RUN" LIMITS

1) Let $f(x) = \frac{1}{x} + 2$.

   a) Draw the graph of $y = \frac{1}{x} + 2$.

   b) Evaluate $\lim_{x \to \infty} \left(\frac{1}{x} + 2 \right)$.

   c) What is the equation of the horizontal asymptote (HA) for the graph in a)?

2) For the following, assume that the graph of a function f is given by $y = f(x)$. (See Parts A and C.)

   a) What are the possible numbers of horizontal asymptotes (HAs) that the graph of a nonconstant polynomial function can have?

   b) What are the possible numbers of HAs that the graph of a rational function can have?

   c) What are the possible numbers of HAs that the graph of a function can have?

LIMIT FORMS

3) In Section 2.4, we will discuss the Limit Form $\frac{1}{0^+}$. Fill out the table below and make a conjecture (guess) as to what this Limit Form yields.

$\frac{1}{1/10}$	$\frac{1}{1/100}$	$\frac{1}{1/1000}$	$\frac{1}{1/10{,}000}$

4) For each of the Limit Forms below, find the limit that it yields.
If 0^+ is appropriate, then write 0^+. If 0^- is appropriate, then write 0^-.
It may help to experiment with sequences of numbers and with extreme numbers.
As a last resort, refer to Section 2.5, Part D.

a) $\frac{3}{\infty}$

b) $\frac{2}{-\infty}$

c) $\frac{-4}{-\infty}$

d) $\frac{\infty}{-2}$

e) $\frac{\infty}{0^+}$

f) $4 \cdot \infty$

g) $\infty - 4$

h) $\infty^{-\infty}$

i) 0^{∞}

j) 3^{∞}

k) $\left(\frac{1}{3}\right)^{\infty}$

5) Yes or No: If $\left[\lim_{x\to\infty} f(x) = 0, \text{ and } \lim_{x\to\infty} g(x) = \infty\right]$, then must it be true that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$?

6) Assume $\lim_{x\to\infty} g(x)$ does not exist (DNE) for a function g.

- If $\lim_{x\to\infty} \frac{g(x)}{h(x)}$ cannot exist for a function h, write "cannot exist" and explain why it cannot exist.

- If $\lim_{x\to\infty} \frac{g(x)}{h(x)}$ might exist, write "might exist" and give an example.

“LONG-RUN” LIMITS

7) Evaluate the following “long-run” limits.

a) $\lim\limits_{x \to -\infty} \tan x$

b) $\lim\limits_{x \to \infty} \frac{4}{x^5}$

c) $\lim\limits_{x \to -\infty} \frac{-3}{x^6}$

d) $\lim\limits_{x \to \infty} \frac{4}{x^{2/3}}$

e) $\lim\limits_{x \to -\infty} \frac{\sqrt{3}}{x^{3/2}}$

f) $\lim\limits_{x \to \infty} \frac{1}{x^{-2}}$

8) Evaluate the following “long-run” limits for polynomial functions.

a) $\lim\limits_{x \to \infty} \sqrt{\pi}$

b) $\lim\limits_{x \to \infty} \left(x^5 + 3x^4 - 2\right)$.

i. Apply a “short cut” using Dominant Term Substitution (“DTS”).
ii. Also give a more rigorous solution using factoring.

c) $\lim\limits_{x \to -\infty} \left(2x^3 - 6x^2 + x\right)$.

i. Apply a “short cut” using Dominant Term Substitution (“DTS”).
ii. Also give a more rigorous solution using factoring.

d) $\lim\limits_{w \to \infty} \left(5w - 4w^4\right)$.

Apply a “short cut” using Dominant Term Substitution (“DTS”).

9) Evaluate the following “long-run” limits for rational functions.

a) Assuming $g(r) = \frac{3r^3 + r - 4}{2r^5 - 7r^2}$, evaluate $\lim\limits_{r \to \infty} g(r)$.

i. Use a “short cut” to figure out the answer quickly.
Explain your answer.

ii. Apply a “short cut” using Dominant Term Substitution (“DTS”).

iii. Also give a more rigorous solution based on one of the methods seen in Examples 13-15. Show work.

iv. Since g is a rational function, what must $\lim\limits_{r \to -\infty} g(r)$ then be?

v. What is the equation of the horizontal asymptote (HA) for the graph of $s = g(r)$ in the rs-plane?

b) $\lim_{x \to -\infty} \dfrac{7x^4 - 5x}{3x^4 + 2}$.

i. Use a "short cut" to figure out the answer quickly. Explain your answer.

ii. Apply a "short cut" using Dominant Term Substitution ("DTS").

iii. Also give a more rigorous solution based on one of the methods seen in Examples 13-15. Show work.

iv. What is the equation of the horizontal asymptote (HA) for the graph of $y = \dfrac{7x^4 - 5x}{3x^4 + 2}$ in the *xy*-plane?

c) $\lim_{x \to \infty} \dfrac{\sqrt{2}x^5 + 11x^8 - \pi}{6x^5 + x}$.

i. Apply a "short cut" using Dominant Term Substitution ("DTS").

ii. Does the graph of $y = \dfrac{\sqrt{2}x^5 + 11x^8 - \pi}{6x^5 + x}$ in the *xy*-plane have a horizontal asymptote (HA)?

iii. What is $\lim_{x \to -\infty} \dfrac{\sqrt{2}x^5 + 11x^8 - \pi}{6x^5 + x}$?

d) $\lim_{t \to \infty} \dfrac{\left(t^3 + 1\right)^2}{4t^6}$.

10) Let $f(x) = \dfrac{-3x^4 + 2x^3 + x^2 - 3x + 2}{x^3 + 1}$.

a) Use Long Division to rewrite $f(x)$ in the form:

(*polynomial*) + (*proper rational expression*). Show work.

b) Evaluate $\lim_{x \to \infty} f(x)$.

c) Evaluate $\lim_{x \to -\infty} f(x)$.

d) What is the equation of the slant asymptote (SA) for the graph of $y = f(x)$ in the *xy*-plane?

11) Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{3x^4+1}}{7x^2}$ by first rewriting $\dfrac{\sqrt{3x^4+1}}{7x^2}$ as $\sqrt{(\text{expression in } x)}$.

12) **ADDITIONAL PROBLEM:** Evaluate the following "long-run" limits for algebraic functions.

a) $\lim_{x\to-\infty} \left(3x^{5/3} - 4x + 2 - x^{-4}\right)$.

b) $\lim_{x\to-\infty} \left(3x^{5/4} - 4x + 2 - x^{-4}\right)$.

c) $\lim_{x\to\infty} \dfrac{\sqrt{4x^6+x^2}+2x^2}{5x^3-\sqrt[3]{x}}$. Use Dominant Term Substitution ("DTS").

d) $\lim_{x\to-\infty} \dfrac{\sqrt{4x^6+x^2}+2x^2}{5x^3-\sqrt[3]{x}}$. Use Dominant Term Substitution ("DTS").

e) Judging from your results in c) and d), what are the horizontal asymptotes (HAs) for the graph of $y = \dfrac{\sqrt{4x^6+x^2}+2x^2}{5x^3-\sqrt[3]{x}}$ in the *xy*-plane?

f) $\lim_{z\to\infty} \dfrac{\sqrt[3]{5z^{12}+7z^7}}{z^5+2}$. Use Dominant Term Substitution ("DTS").

g) $\lim_{x\to\infty} \left(\sqrt{x^4+5x^2} - x^2\right)$.

ADDITIONAL PROBLEM: A WORD PROBLEM

13) (Bacterial populations). At midnight, a large petri dish contains 2500 bacteria of the species *E. calculi*. Starting at midnight, a stream is poured into the petri dish that adds 100 *E. calculi* bacteria and 150 *E. coli* bacteria (and no other bacteria) to the dish every second.

a) Find an expression for $p(t)$, the proportion of the bacteria in the petri dish that are *E. calculi* t seconds after midnight, where $t \geq 0$.

b) Find $\lim_{t\to\infty} p(t)$, and interpret the result. Discuss the realism of this problem.

SECTION 2.4: LIMITS AND INFINITY II

VERTICAL ASYMPTOTES (VAs) and INFINITE LIMITS AT A POINT

1) Let $f(x) = \dfrac{1}{x-3}$.

   a) Draw the graph of $y = \dfrac{1}{x-3}$.

   b) Evaluate $\lim\limits_{x \to 3^+} \dfrac{1}{x-3}$.

   c) Evaluate $\lim\limits_{x \to 3^-} \dfrac{1}{x-3}$.

   d) What is the equation of the vertical asymptote (VA) for the graph in a)?

2) For the following, assume that the graph of a function f is given by $y = f(x)$.

   a) What are the possible numbers of vertical asymptotes (VAs) that the graph of a polynomial function can have?

   b) What are the possible numbers of VAs that the graph of a rational function can have?

   c) What are the possible numbers of VAs that the graph of a function can have?

LIMIT FORMS

3) For each of the Limit Forms below, find the limit that it yields.

   a) $\dfrac{5}{0^+}$

   b) $\dfrac{-3}{0^+}$

   c) $\dfrac{\pi}{0^-}$

   d) $\dfrac{-\sqrt{2}}{0^-}$

RATIONAL FUNCTIONS

4) Consider $f(x) = \dfrac{3x-2}{x^3 - 3x^2 + 4}$.

a) Factor the denominator. You may want to review Section 2.3 on the Rational Zero Test (Rational Roots Theorem) and Synthetic Division in the Precalculus notes. Show work.

b) Evaluate the following limits at a point. Show work.

i. $\lim_{x \to -1^+} f(x)$

ii. $\lim_{x \to -1^-} f(x)$

iii. $\lim_{x \to -1} f(x)$

iv. $\lim_{x \to 2^+} f(x)$

v. $\lim_{x \to 2^-} f(x)$

vi. $\lim_{x \to 2} f(x)$

vii. $\lim_{x \to 0} f(x)$

c) Evaluate the following "long-run" limits. You may use "short cuts."

i. $\lim_{x \to \infty} f(x)$

ii. $\lim_{x \to -\infty} f(x)$

d) What is the equation of the horizontal asymptote (HA) for the graph of $y = f(x)$ in the *xy*-plane?

e) What are the equations of the vertical asymptotes (VAs)?

f) What is the *x*-intercept of the graph?

g) What is the *y*-intercept of the graph?

h) Based on your results in a) through g), sketch a guess as to what the graph of $y = f(x)$ should look like.

5) Give the rule $f(x)$ for a rational function f such that the graph of $y = f(x)$ in the *xy*-plane has a horizontal asymptote (HA) at $y = 4$ and vertical asymptotes (VAs) at $x = -2$ and $x = 3$.

6) Consider $g(t) = \dfrac{t - 6t^2}{2t^2 - 8t + 6}$.

   a) Find the equations of the vertical asymptotes (VAs) of the graph of $w = g(t)$ in the *tw*-plane. Justify your answer using limits. Show work.

   b) Find the equation of the horizontal asymptote (HA) of the graph of $w = g(t)$. Justify your answer using limits. Show work by using a rigorous method from Section 2.3.

7) Consider $h(z) = \dfrac{z^4 - 3z + 2}{z^2 + 1}$. How many vertical asymptotes (VAs) and horizontal asymptotes (HAs) does the graph of $p = h(z)$ have in the *zp*-plane?

OTHER EXAMPLES

8) Evaluate the following limits at a point.

   a) $\displaystyle\lim_{\theta \to \left(\frac{\pi}{2}\right)^-} \tan\theta$

   b) $\displaystyle\lim_{\theta \to \left(\frac{\pi}{2}\right)^+} \tan\theta$

   c) $\displaystyle\lim_{x \to 0^+} \csc x$

   d) $\displaystyle\lim_{x \to \pi^-} \csc x$

9) (Einstein's Theory of Relativity). A particular object at rest has mass m_0 (measured in kilograms, let's say). The speed of light in a vacuum, denoted by c in physics, is about 186,282 miles per second, or exactly 299,792,458 meters per second; the meter is now defined as a consequence of this. If the object is traveling with speed v, then the mass of the object is given by: m, or $m(v) = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

   a) Evaluate $\displaystyle\lim_{v \to c^-} m(v)$, and interpret the result.

   b) Discuss $\displaystyle\lim_{v \to c^+} m(v)$.

See Michael Fowler's web page on relativistic mass increase:
http://galileo.phys.virginia.edu/classes/109N/lectures/mass_increase.html

SECTION 2.5 : THE INDETERMINATE FORMS $\frac{0}{0}$ AND $\frac{\infty}{\infty}$

1) Give function rules for $f(x)$ and $g(x)$ such that $\lim_{x\to\infty} f(x)=\infty$ and $\lim_{x\to\infty} g(x)=\infty$, but $\lim_{x\to\infty} \frac{f(x)}{g(x)}=0$.

2) Same as 1), but give rules such that $\lim_{x\to\infty} \frac{f(x)}{g(x)}=\infty$.

3) Evaluate the following limits of the form 0/0 at a point. Show work.

a) $\lim_{x\to 3} \frac{x^2-9}{x-3}$

b) $\lim_{r\to 6} \frac{3r^2-17r-6}{36-r^2}$

c) $\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25}$

d) $\lim_{t\to 5} \frac{\sqrt{11+t}-4}{t-5}$

e) $\lim_{\theta\to -2} \frac{\frac{1}{\theta}+\frac{1}{2}}{\theta+2}$

f) $\lim_{x\to \frac{1}{2}} \frac{2x^3-x^2+8x-4}{2x-1}$.
(Various methods can be used.)

g) $\lim_{x\to 2} \frac{x^3-8}{x-2}$

4) Sketch a guess as to what the graph of $f(x)=\frac{x^2-2x-3}{x^2-3x-4}$ should look like.

a) Find the domain of f.

b) Find the x-intercept(s), if any.

c) Find the y-intercept, if any.

d) Identify whether f is even, odd, or neither.

e) Write the equations of any asymptotes for the graph. Justify using limits.

f) Find any holes "on" the graph. Justify using limits.

g) Sketch your graph. Incorporate all of the above in your sketch.

SECTION 2.6: THE SQUEEZE (SANDWICH) THEOREM

1) Find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right)$, and prove it.

2) Find $\lim_{t\to 0} \left(t^4 + \sin^2 t\right)\cos\left(\frac{t+3}{\sqrt[3]{t^2 - t}}\right)$, and prove it.

3) Assume that there exist real constants c and d such that $c \le f(x) \le d$ for all real values of x (except possibly at 0). Find $\lim_{x\to 0} x^8 f(x)$, and prove it.

4) Find $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)$, and prove it. **(← ADDITIONAL PROBLEM)**

5) Find $\lim_{x\to\infty} \frac{\cos x}{x^5}$, and prove it.

6) Find $\lim_{\theta\to-\infty} \frac{5\sin(3\theta)}{4\theta^3}$, and prove it.

7) Evaluate $\lim_{x\to\infty} \frac{5x + \sin x}{x}$.

8) Refer back to Exercise 1. Is it true that

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right) = \left[\lim_{x\to 0} x^2\right]\left[\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)\right]?$$

Why does this <u>not</u> contradict the list of properties at the beginning of Section 2.2?

SECTION 2.7: PRECISE DEFINITIONS OF LIMITS

ADDITIONAL PROBLEMS (#1-7)

1) Use the ε-δ definition of $\lim_{x\to a} f(x) = L$ to prove that $\lim_{x\to 2}(3x-7) = -1$.

2) In Exercise 1, given that $\varepsilon = 0.6$, find the largest value of δ such that, if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$. Answer the same question for $\varepsilon = 0.06$ and $\varepsilon = 0.006$.

3) Use the ε-δ definition of $\lim_{x\to a} f(x) = L$ to prove that $\lim_{x\to -8}\left(5+\frac{1}{4}x\right) = 3$.

4) Consider the statement $\lim_{x\to 4} 6 = 6$, which is of the form $\lim_{x\to a} f(x) = L$. For any positive real value of ε, what are the positive real values of δ such that, if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$?

5) The following statements are of the form $\lim_{x\to a} f(x) = L$. For the given value of ε, find the largest value of δ such that, if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$. Graphs may help.

   a) $\lim_{x\to 9}\sqrt{x} = 3$, given $\varepsilon = 0.1$; give an exact answer.

   b) $\lim_{x\to 2} x^3 = 8$, given $\varepsilon = 0.01$; round off your answer to five significant figures.

6) Give a precise "M-δ" definition of $\lim_{x\to a} f(x) = \infty$, where *a* is a real constant, and the function *f* is defined on an open interval containing *a*, possibly excluding *a* itself.

7) Give a precise "N-δ" definition of $\lim_{x\to a} f(x) = -\infty$, where *a* is a real constant, and the function *f* is defined on an open interval containing *a*, possibly excluding *a* itself.

KNOW THE FOLLOWING

• Precise ε-δ definition of $\lim_{x \to a} f(x) = L$.

• We will not have time to cover the precise definitions of:

$$\lim_{x \to a^+} f(x) = L \qquad \lim_{x \to \infty} f(x) = L \qquad \lim_{x \to a} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to -\infty} f(x) = L \qquad \lim_{x \to a} f(x) = -\infty$$

SECTION 2.8: CONTINUITY

CLASSIFYING DISCONTINUITIES

1) For each of the following, find all discontinuities, classify them by using limits, give the continuity interval(s) for the corresponding function, and graph the function.

   a) $g(t) = t^2 - 4t + 3$

   b) $f(x) = \dfrac{x^2 - 5x + 6}{x - 2}$

   c) $h(r) = \dfrac{4r + 12}{r^2 + 6r + 9}$

   d) $f(x) = \begin{cases} x^2 + x, & \text{if } x \le 2 \\ 6 - x, & \text{if } x > 2 \end{cases}$

   e) $g(x) = \begin{cases} x^2 + x, & \text{if } x < 2 \\ 8 - x, & \text{if } x > 2 \end{cases}$ (Variation on d))

   f) $h(x) = \begin{cases} x^2 + x, & \text{if } x < 2 \\ 8 - x, & \text{if } x \ge 2 \end{cases}$ (Variation on d) and e))

   On f), discuss the continuity of h at 2, and justify your conclusion.

   g) $f(x) = \dfrac{\left|x^2 - 4\right|}{x^2 - 4}$

2) Your bank account is accruing continuously compounded interest. At noon today, you withdraw \$200 from the account. If the amount of money in your account is plotted against time, what type of discontinuity appears at noon?

CONTINUITY

3) Draw a graph where f is defined on $[a, b]$, and f is continuous on (a, b), but f is not continuous on the closed interval $[a, b]$.

4) Determine A such that the function f defined below is continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & \text{if } x \neq 5 \\ A, & \text{if } x = 5 \end{cases}$$

5) Let $f(x) = \dfrac{\sqrt{x-1} + \sqrt[3]{x+5}}{x^2 - 7x + 12}$. What are the continuity intervals of f?

6) Let $h(x) = \csc\left(\dfrac{1}{\sqrt{x}}\right)$. Where is h continuous? Show work!

THE INTERMEDIATE VALUE THEOREM (IVT)

7) a) Use the IVT to prove that the following equation has a solution between 1 and 2: $3x^3 - 2x^2 - 2x - 5 = 0$.

b) Use Synthetic Division (see Section 2.3 in the Precalculus notes) to show that $\dfrac{5}{3}$ is such a solution.

8) The height of a projectile t seconds after it is fired is given by $s(t) = -16t^2 + 30t + 4$ in feet, where $0 \leq t \leq 2$.

a) Use the IVT to prove that the projectile achieves a height of 15 feet sometime within one second after being fired.

b) Find the value for t at which this happens. Show work!

ADDITIONAL PROBLEMS (#9-10)

9) Verify the IVT for $f(x) = x^2 + 5$ on the x-interval $[1, 3]$.

10) Verify the IVT for $f(x) = x^2 + 4x - 1$ on the x-interval $[-1, 2]$.

KNOW THE FOLLOWING

- The definitions of continuity of a function at a point and on an open interval.

- Recognize continuity on a closed interval.

CALCULUS: THE ANSWERS

MATH 150: CALCULUS WITH ANALYTIC GEOMETRY I

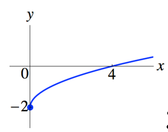
VERSION 1.3

KEN KUNIYUKI and LALEH HOWARD
SAN DIEGO MESA COLLEGE

CHAPTER 1: REVIEW

1) $f(-4)=16$; $f(a+h)=(a+h)^2=a^2+2ah+h^2$
2) Polynomial: Yes, Rational: Yes, Algebraic: Yes
3) Polynomial: No, Rational: Yes, Algebraic: Yes
4) Polynomial: No, Rational: No, Algebraic: Yes
5) Polynomial: No, Rational: No, Algebraic: No
6) a) $(-\infty,\infty)$, b) $(-\infty,\infty)$, c) $\left(-\infty,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},2\right)\cup(2,\infty)$, d) $[-2,5)\cup(5,\infty)$,

   e) $(-2,\infty)$, f) $(-\infty,7]$, g) $(-\infty,-1]\cup[3,\infty)$
7)



; Domain is $[0,\infty)$; Range is $[-2,\infty)$

8) the y-axis; the function is even
9) the origin; the function is odd
10) the function is even
11) the function is even
12) the function is neither even nor odd
13) $(f\circ g)(x)=x^8+2x^6+x^4+\dfrac{1}{x^4+x^2+1}$. $\mathrm{Dom}(f\circ g)=\mathbb{R}$, or $(-\infty,\infty)$.
14) $g(x)=x^4+x$, $f(u)=u^8$; there are other possibilities
15) $g(t)=\dfrac{1}{t}$, $f(u)=\sqrt[4]{u}$; there are other possibilities
16) $g(r)=r^2$, $f(u)=\sin u$; there are other possibilities
17) a) undefined, b) -1, c) $\sqrt{2}$, d) $-\dfrac{2\sqrt{3}}{3}$, e) $-\dfrac{\sqrt{3}}{2}$, f) $-\sqrt{3}$
18) Hint on a): Use a Double-Angle ID; Hint on b): Use a Pythagorean ID
19) a) $\left\{x\in\mathbb{R}\,\middle|\,x=-\dfrac{\pi}{6}+2\pi n,\text{ or } x=\dfrac{7\pi}{6}+2\pi n,\text{ or } x=\dfrac{3\pi}{2}+2\pi n\quad(n\in\mathbb{Z})\right\}$

   Note: $-\dfrac{\pi}{6}$ can be replaced by $\dfrac{11\pi}{6}$, $\dfrac{3\pi}{2}$ can be replaced by $-\dfrac{\pi}{2}$, etc.

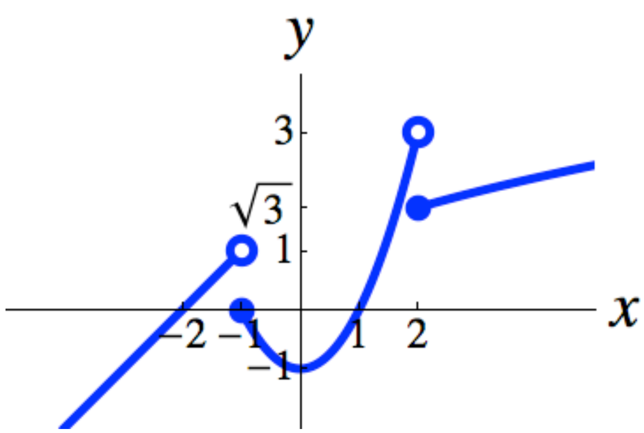
   b) $\left\{x\in\mathbb{R}\,\middle|\,x=\pm\dfrac{\pi}{9}+\dfrac{2\pi n}{3}\quad(n\in\mathbb{Z})\right\}$, or, equivalently,

   $\left\{x\in\mathbb{R}\,\middle|\,x=\dfrac{\pi}{9}+\dfrac{2\pi n}{3},\text{ or } x=\dfrac{5\pi}{9}+\dfrac{2\pi n}{3}\quad(n\in\mathbb{Z})\right\}$

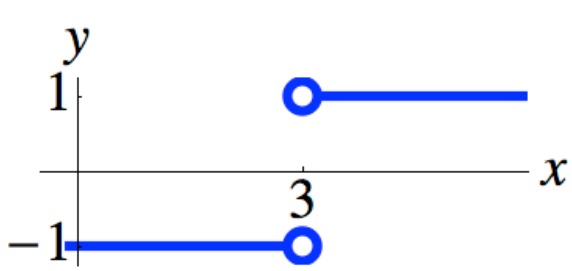
CHAPTER 2: LIMITS AND CONTINUITY

SECTION 2.1: AN INTRODUCTION TO LIMITS

1) 57
2) 11/7
3) 10
4) π^2
5) a) $f(1) = 0.9999$, $f(0.1) = 0.0999$, $f(0.01) = 0.0099$; b) -0.0001, No
6) a) 11/7, b) 11/7; as a consequence, the answer to Exercise 2 is the same.
7) No; a counterexample: $\lim_{x \to 0^+} \sqrt{x} = 0$, while $\lim_{x \to 0} \sqrt{x}$ does not exist (DNE).
   See also Example 6.
8) Yes
9) No; a counterexample: see Example 8 on $h(x) = \begin{cases} x+3, & x \neq 3 \\ 7, & x = 3 \end{cases}$
10) "might exist"; for example, see Example 7 on $g(x) = x + 3,\ (x \neq 3)$.
11) a) b) 1, 0, DNE; c) 3, $\sqrt{3}$, DNE, 2



12) a) b) -1, 1, DNE



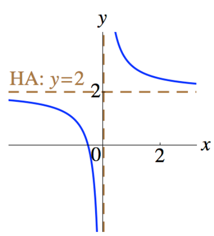
13) a) 0 (liters), which means that, if the gas's temperature approaches absolute zero (from above), its volume approaches zero (liters).
    b) DNE, because temperatures cannot go below absolute zero. The domain of V does not include values of T below absolute zero.

SECTION 2.2: PROPERTIES OF LIMITS and ALGEBRAIC FUNCTIONS

1) $\dfrac{39+18\sqrt{3}}{16}$, or $\dfrac{3}{16}\left(13+6\sqrt{3}\right)$
2) a) DNE, b) 0, c) DNE
3) a) 0, b) 0, c) 0
4) a) 0, b) DNE, c) DNE, d) 0, e) $\sqrt{5}$, f) DNE, g) 0
5) a) DNE, b) 0, c) DNE, d) 0, e) DNE, f) DNE, g) $\sqrt{6}$
6) Yes, by linearity of the limit operator.
7) "might exist"; for example, $\lim\limits_{x\to 0}\sqrt{x^2}=0$. See also Example 7 on $\lim\limits_{x\to -7}\sqrt{(x+7)^2}$.

SECTION 2.3: LIMITS AND INFINITY I

1) a) b) 2, c) $y=2$



2) a) 0; b) 0 or 1; c) 0, 1, or 2
3) In the table: 10 100 1000 10,000. The Limit Form yields ∞.
4) a) 0^+, b) 0^-, c) 0^+, d) $-\infty$, e) ∞, f) ∞, g) ∞, h) 0^+, i) 0, j) ∞, k) 0^+
5) Yes
6) "might exist"; for example, Example 6 on $f(x)=\dfrac{\sin x}{x}$; also, $f(x)=\dfrac{\sin x+2}{\sin x+2}$.
7) a) DNE, b) 0 or 0^+, c) 0 or 0^-, d) 0 or 0^+, e) DNE, f) ∞
8) a) $\sqrt{\pi}$

   b) i. $\lim\limits_{x\to\infty}\left(x^5+3x^4-2\right)=\lim\limits_{x\to\infty}x^5=\infty$;

   ii. $\lim\limits_{x\to\infty}\left(x^5+3x^4-2\right)=\lim\limits_{x\to\infty}\underbrace{x^5}_{\to\infty}\underbrace{\left(1+\dfrac{3}{x}-\dfrac{2}{x^5}\right)}_{\to 1}=\infty$

   c) i. $\lim\limits_{x\to-\infty}\left(2x^3-6x^2+x\right)=\lim\limits_{x\to-\infty}2x^3=-\infty$;

   ii. $\lim\limits_{x\to-\infty}\left(2x^3-6x^2+x\right)=\lim\limits_{x\to-\infty}\underbrace{2x^3}_{\to-\infty}\underbrace{\left(1-\dfrac{3}{x}+\dfrac{1}{2x^2}\right)}_{\to 1}=-\infty$

   d) $\lim\limits_{w\to\infty}\left(5w-4w^4\right)=\lim\limits_{w\to\infty}\left(-4w^4\right)=-\infty$

9) a) i. 0, because g is a proper rational function, and we seek a "long-run" limit;

ii. $\lim_{r\to\infty} g(r) = \lim_{r\to\infty} \frac{3r^3}{2r^5} = \lim_{r\to\infty} \frac{3}{2r^2} = 0$;

iii. $$\lim_{r\to\infty} g(r) = \lim_{r\to\infty} \frac{\frac{3r^3}{r^5} + \frac{r}{r^5} - \frac{4}{r^5}}{\frac{2r^5}{r^5} - \frac{7r^2}{r^5}} = \lim_{r\to\infty} \frac{\overbrace{\frac{3}{r^2}}^{\to 0} + \overbrace{\frac{1}{r^4}}^{\to 0} - \overbrace{\frac{4}{r^5}}^{\to 0}}{2 - \underbrace{\frac{7}{r^3}}_{\to 0}} = \frac{0}{2} = 0, \text{ or}$$

$$\lim_{r\to\infty} g(r) = \lim_{r\to\infty} \frac{3r^3 \overbrace{\left(1 + \frac{1}{3r^2} - \frac{4}{3r^3}\right)}^{\to 1}}{2r^5 \underbrace{\left(1 - \frac{7}{2r^3}\right)}_{\to 1}} = \lim_{r\to\infty} \frac{3r^3}{2r^5} = \lim_{r\to\infty} \frac{3}{2r^2} = 0;$$

iv. $\lim_{r\to-\infty} g(r) = 0$, also;

v. HA: $s = 0$

b) i. $\frac{7}{3}$, because the numerator and the denominator have the same degree, and we seek a "long-run" limit, so we take the ratio of the leading coefficients;

ii. $\lim_{x\to-\infty} \frac{7x^4 - 5x}{3x^4 + 2} = \lim_{x\to-\infty} \frac{7x^4}{3x^4} = \frac{7}{3}$;

iii. $$\lim_{x\to-\infty} \frac{7x^4 - 5x}{3x^4 + 2} = \lim_{x\to-\infty} \frac{\frac{7x^4}{x^4} - \frac{5x}{x^4}}{\frac{3x^4}{x^4} + \frac{2}{x^4}} = \lim_{x\to-\infty} \frac{7 - \overbrace{\frac{5}{x^3}}^{\to 0}}{3 + \underbrace{\frac{2}{x^4}}_{\to 0}} = \frac{7}{3}, \text{ or}$$

$$\lim_{x\to-\infty} \frac{7x^4 - 5x}{3x^4 + 2} = \lim_{x\to-\infty} \frac{7x^4 \overbrace{\left(1 - \frac{5}{7x^3}\right)}^{\to 1}}{3x^4 \underbrace{\left(1 + \frac{2}{3x^4}\right)}_{\to 1}} = \lim_{x\to-\infty} \frac{7x^4}{3x^4} = \frac{7}{3};$$

iv. HA: $y = \frac{7}{3}$

c) i. $\lim\limits_{x\to\infty} \dfrac{\sqrt{2}x^5 + 11x^8 - \pi}{6x^5 + x} = \lim\limits_{x\to\infty} \dfrac{11x^8}{6x^5} = \lim\limits_{x\to\infty} \dfrac{11}{6}x^3 = \infty$;

ii. No, the graph has no HA;

iii. $-\infty$

d) 1/4

10) a) $f(x) = -3x + 2 + \dfrac{x^2}{x^3+1}$; b) $-\infty$; c) ∞; d) $y = -3x + 2$, or $y = 2 - 3x$

11) $\dfrac{\sqrt{3}}{7}$

12) a) $-\infty$

b) DNE

c) $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^6 + x^2} + 2x^2}{5x^3 - \sqrt[3]{x}} = \lim\limits_{x\to\infty} \dfrac{\sqrt{4x^6} + 2x^2}{5x^3} = \lim\limits_{x\to\infty} \dfrac{2\left|x^3\right| + 2x^2}{5x^3} = \lim\limits_{x\to\infty} \dfrac{2x^3}{5x^3} = \dfrac{2}{5}$

d)

$$\lim_{x\to-\infty} \frac{\sqrt{4x^6 + x^2} + 2x^2}{5x^3 - \sqrt[3]{x}} = \lim_{x\to-\infty} \frac{\sqrt{4x^6} + 2x^2}{5x^3} = \lim_{x\to-\infty} \frac{2\left|x^3\right| + 2x^2}{5x^3}$$

$$= \lim_{x\to-\infty} \frac{-2x^3}{5x^3} = -\frac{2}{5}$$

e) HAs: $y = \dfrac{2}{5}$ and $y = -\dfrac{2}{5}$

f) $\lim\limits_{z\to\infty} \dfrac{\sqrt[3]{5z^{12} + 7z^7}}{z^5 + 2} = \lim\limits_{z\to\infty} \dfrac{\sqrt[3]{5z^{12}}}{z^5} = \lim\limits_{z\to\infty} \dfrac{\sqrt[3]{5}z^4}{z^5} = \lim\limits_{z\to\infty} \dfrac{\sqrt[3]{5}}{z} = 0$

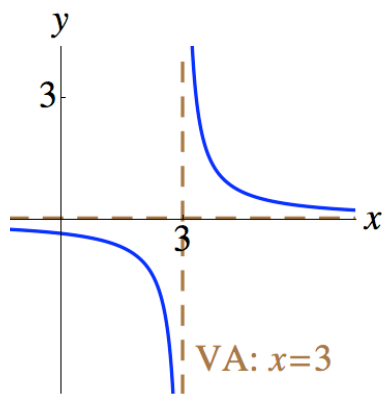
g) 5/2. Hint: Rationalize as in Example 19.

13) a) $p(t) = \dfrac{2500 + 100t}{2500 + 250t} \left(\dfrac{E.\ calculi \text{ bacteria in the dish}}{\text{total bacteria in the dish}} \right)$;

b) $\dfrac{2}{5} \left(\dfrac{E.\ calculi \text{ bacteria in the dish}}{\text{total bacteria in the dish}} \right)$; the proportion of the bacteria in the Petri dish that are *E. calculi* approaches $\dfrac{2}{5}$ in the long run, the same as for the incoming stream; this calculation assumes that the petri dish has infinite capacity and that infinitely many bacteria are available, which is unrealistic.

SECTION 2.4: LIMITS AND INFINITY II

1) a)



b) ∞, c) $-\infty$, d) $x=3$

2) a) 0
b) any nonnegative integer number
c) any nonnegative integer number, or infinitely many

3) a) ∞, b) $-\infty$, c) $-\infty$, d) ∞

4) a) $f(x)=\dfrac{3x-2}{(x+1)(x-2)^2}$

b) i. $-\infty$. Work: $\lim\limits_{x\to-1^+} f(x)=\lim\limits_{x\to-1^+} \dfrac{\overbrace{3x-2}^{\to -5}}{\underbrace{(x+1)}_{\to 0^+}\underbrace{(x-2)^2}_{\to 9}}$ $\left(\text{Limit Form } \dfrac{-5}{0^+}\right)=-\infty$.

ii. ∞, iii. DNE, iv. ∞, v. ∞, vi. ∞, vii. $-\dfrac{1}{2}$

c) i. 0, ii. 0

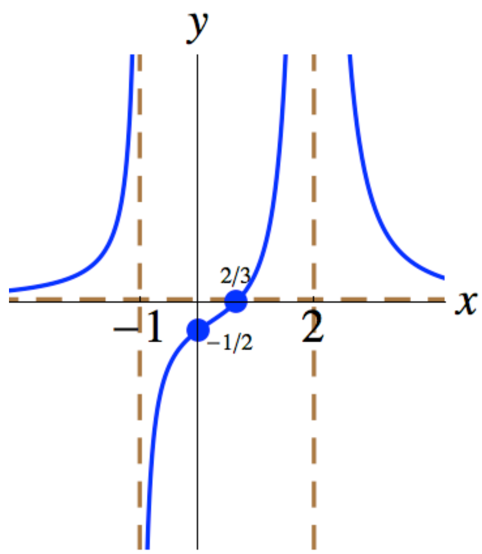
d) HA: $y=0$

e) VAs: $x=-1$ and $x=2$

f) $\dfrac{2}{3}$, or $\left(\dfrac{2}{3},0\right)$

g) $-\dfrac{1}{2}$, or $\left(0,-\dfrac{1}{2}\right)$

h)



5) $f(x)=\dfrac{4x^2}{(x+2)(x-3)}$; there are other possibilities

6) a) VAs: $t=1$ and $t=3$;

for $t=1$, it is sufficient to show either $\lim_{t\to 1^+} g(t)=\infty$, or $\lim_{t\to 1^-} g(t)=-\infty$;

for $t=3$, it is sufficient to show either $\lim_{t\to 3^+} g(t)=-\infty$, or $\lim_{t\to 3^-} g(t)=\infty$.

b) HA: $w=-3$;

$$\lim_{t\to\infty} g(t)=\lim_{t\to\infty}\frac{\dfrac{t}{t^2}-\dfrac{6t^2}{t^2}}{\dfrac{2t^2}{t^2}-\dfrac{8t}{t^2}+\dfrac{6}{t^2}}=\lim_{t\to\infty}\frac{\overbrace{\dfrac{1}{t}}^{\to 0}-6}{2-\underbrace{\dfrac{8}{t}}_{\to 0}+\underbrace{\dfrac{6}{t^2}}_{\to 0}}=\frac{-6}{2}=-3;$$

also, $\lim_{t\to -\infty} g(t)=-3$ (observe that g is a rational function).

7) 0 VAs, 0 HAs

8) a) ∞, b) $-\infty$, c) ∞, d) ∞

9) a) ∞, which means that, if an object's speed approaches the speed of light (from below), its mass increases without bound

b) DNE, which makes sense because faster-than-light speed is impossible.

SECTION 2.5 : THE INDETERMINATE FORMS $\frac{0}{0}$ AND $\frac{\infty}{\infty}$

1) $f(x)=x$, $g(x)=x^2$; there are other possibilities.

2) $f(x)=x^2$, $g(x)=x$; there are other possibilities.

3) a) 6; see the equivalent function in Section 2.1, Example 9: $g(x)=x+3,\ (x\neq 3)$.

b) $-\frac{19}{12}$, c) $\frac{1}{10}$; Hint: Factor the denominator, or rationalize the numerator.

d) $\frac{1}{8}$, e) $-\frac{1}{4}$

f) $\frac{17}{4}$; Hint: Factor the numerator by grouping, use Long Division, or use the Rational Zero Test (Rational Roots Theorem) and Synthetic Division from Section 2.3 of the Precalculus notes – observe: $2\left(x-\frac{1}{2}\right)=2x-1$.

g) 12; Hint: Factor the numerator.

4) a) $\text{Dom}(f)=\{x\in\mathbb{R}\mid x\neq -1 \text{ and } x\neq 4\}=(-\infty,-1)\cup(-1,4)\cup(4,\infty)$

b) 3, or $(3,0)$

c) $\frac{3}{4}$, or $\left(0,\frac{3}{4}\right)$

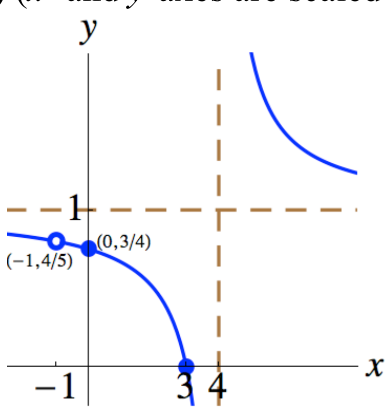
d) neither even nor odd

e) VA: $x=4$, because $\lim\limits_{x\to 4^+} f(x)=\infty$; also, $\lim\limits_{x\to 4^-} f(x)=-\infty$.

HA: $y=1$, because $\lim\limits_{x\to\infty} f(x)=1$.

f) $\left(-1,\frac{4}{5}\right)$, because $\lim\limits_{x\to -1} f(x)=\frac{4}{5}$, and -1 is not in $\text{Dom}(f)$.

g) (x- and y-axes are scaled differently below)



SECTION 2.6: THE SQUEEZE (SANDWICH) THEOREM

1) Shorthand:

$$\text{As } x \to 0, \quad \underbrace{-x^2}_{\to 0} \le \underbrace{x^2 \sin\left(\frac{1}{x^2}\right)}_{\substack{\text{Therefore,} \\ \to 0}} \le \underbrace{x^2}_{\to 0} \quad (\forall x \ne 0)$$

2) Note 1: The domain of $\cos\left(\dfrac{t+3}{\sqrt[3]{t^2 - t}}\right)$ is $\left\{t \in \mathbb{R} \,\middle|\, t \ne 0 \text{ and } t \ne 1\right\}$.

We can, for example, restrict our attention to t-values in $(-1, 1) \setminus \{0\}$, a punctured neighborhood of 0.

Note 2: $t^4 + \sin^2 t > 0$ whenever $t \ne 0$.
Shorthand:

$$\text{As } t \to 0, \quad \underbrace{-\left(t^4 + \sin^2 t\right)}_{\to 0} \le \underbrace{\left(t^4 + \sin^2 t\right)\cos\left(\frac{t+3}{\sqrt[3]{t^2 - t}}\right)}_{\substack{\text{Therefore,} \\ \to 0}} \le \underbrace{t^4 + \sin^2 t}_{\to 0} \quad (\forall t \ne 0, 1)$$

3) Shorthand:

$$\text{As } x \to 0, \quad \underbrace{cx^8}_{\to 0} \le \underbrace{x^8 f(x)}_{\substack{\text{Therefore,} \\ \to 0}} \le \underbrace{dx^8}_{\to 0} \quad (\forall x \ne 0)$$

4) Shorthand:

$$\text{As } x \to 0, \quad \underbrace{-|x|}_{\to 0} \le \underbrace{x \cos\left(\frac{1}{x}\right)}_{\substack{\text{Therefore,} \\ \to 0}} \le \underbrace{|x|}_{\to 0} \quad (\forall x \ne 0)$$

5) Shorthand:

$$\text{As } x \to \infty, \quad \underbrace{-\frac{1}{x^5}}_{\to 0} \le \underbrace{\frac{\cos x}{x^5}}_{\substack{\text{Therefore,} \\ \to 0}} \le \underbrace{\frac{1}{x^5}}_{\to 0} \quad (\forall x > 0)$$

6) Note: If $\theta < 0$, then $4\theta^3 < 0$, and $-4\theta^3 > 0$.
Shorthand:

$$\text{As } \theta \to -\infty, \quad \underbrace{\frac{5}{4\theta^3}}_{\to 0} \le \underbrace{\frac{5\sin(3\theta)}{4\theta^3}}_{\substack{\text{Therefore,} \\ \to 0}} \le \underbrace{-\frac{5}{4\theta^3}}_{\to 0} \quad (\forall \theta < 0)$$

7) 5.

8) No. The properties listed in Section 2.2, Part A are claimed to be true under the assumption that all of the indicated limits exist as real constants.
Here, $\lim\limits_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ does not exist (DNE).

SECTION 2.7: PRECISE DEFINITIONS OF LIMITS

1) $\left| f(x) - L \right| = \left| (3x-7) - (-1) \right| = \left| 3x - 6 \right| = 3\left| x - 2 \right|$;

$\left| f(x) - L \right| < \varepsilon \quad \Leftrightarrow \quad 3\left| x-2 \right| < \varepsilon \quad \Leftrightarrow \quad \left| x - 2 \right| < \frac{\varepsilon}{3}$; choose $\delta = \frac{\varepsilon}{3}$;

$0 < \left| x - a \right| < \delta \Rightarrow 0 < \left| x - 2 \right| < \frac{\varepsilon}{3} \Rightarrow 0 < 3\left| x - 2 \right| < \varepsilon \Rightarrow \left| f(x) - L \right| < \varepsilon$. Q.E.D.

2) $\delta = 0.2$, $\delta = 0.02$, $\delta = 0.002$

3) Hints: $\left| \frac{1}{4}x + 2 \right| = \frac{1}{4}\left| x - (-8) \right|$; choose $\delta = 4\varepsilon$.

4) All positive real numbers

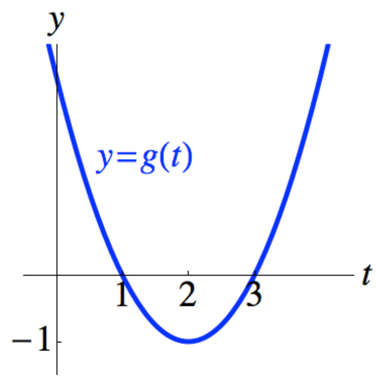
5) a) $9 - (2.9)^2 = 0.59$, b) $\sqrt[3]{8.01} - 2 \approx 0.00083299$

6) $\lim\limits_{x \to a} f(x) = \infty \Leftrightarrow \forall M \in \mathbb{R}, \exists \delta > 0 \ni \left[0 < \left| x - a \right| < \delta \Rightarrow f(x) > M \right]$.

7) $\lim\limits_{x \to a} f(x) = -\infty \Leftrightarrow \forall N \in \mathbb{R}, \exists \delta > 0 \ni \left[0 < \left| x - a \right| < \delta \Rightarrow f(x) < N \right]$.

SECTION 2.8: CONTINUITY

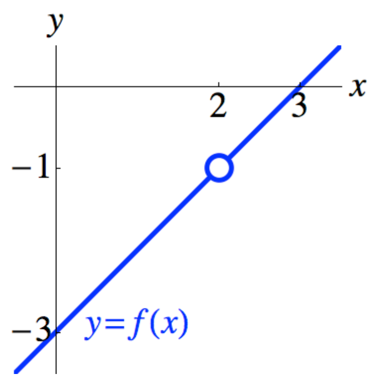
1) a) Discontinuities: None; Continuous on $(-\infty, \infty)$;



b) Observe: $f(x) = x - 3 \quad (x \neq 2)$;

Discontinuities: 2 (removable: $\lim_{x \to 2} f(x) = -1$, but f is undefined at 2);

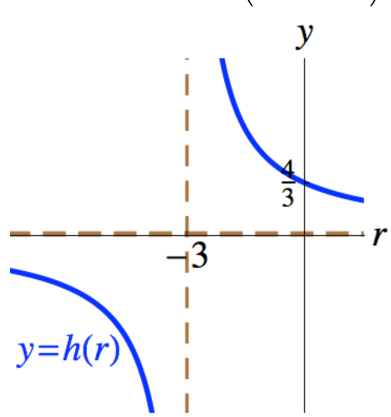
Continuous on $(-\infty, 2), (2, \infty)$;



c) Observe: $h(r) = \dfrac{4}{r+3}$;

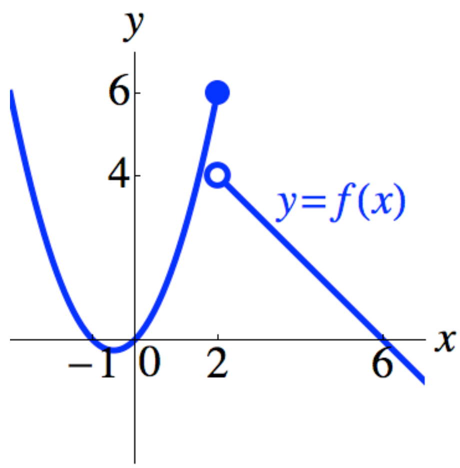
Discontinuities: -3 (infinite: $\lim_{r \to -3^+} h(r) = \infty$; also, $\lim_{r \to -3^-} h(r) = -\infty$);

Continuous on $(-\infty, -3), (-3, \infty)$;

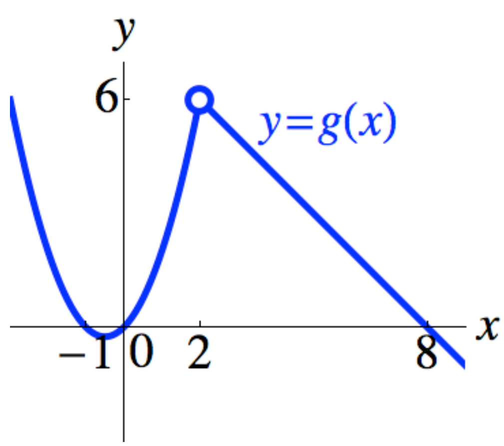


d) Discontinuities: 2 (jump: $\lim_{x\to 2^-} f(x) = 6$, and $\lim_{x\to 2^+} f(x) = 4$, but $6 \neq 4$);

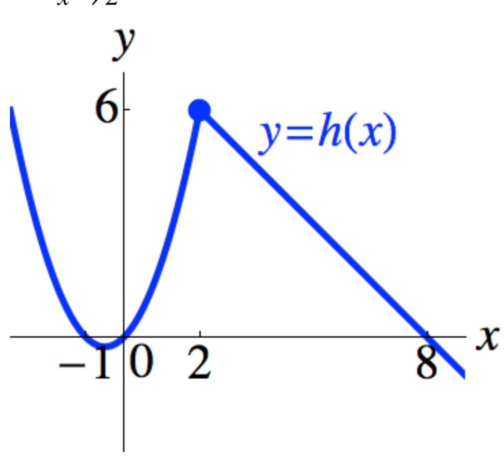
Continuous on $(-\infty, 2], (2, \infty)$;



e) Discontinuities: 2 (removable: $\lim_{x\to 2^-} g(x) = 6$, and $\lim_{x\to 2^+} g(x) = 6$, so $\lim_{x\to 2} g(x) = 6$, but g is undefined at 2); Continuous on $(-\infty, 2), (2, \infty)$;



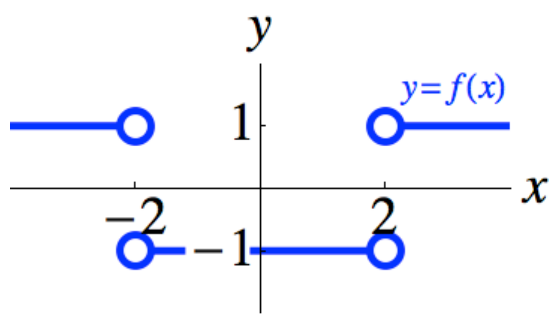
f) Discontinuities: None; h is continuous at 2: $\lim_{x\to 2^-} h(x) = 6$, and $\lim_{x\to 2^+} h(x) = 6$, so $\lim_{x\to 2} h(x) = 6$, and $h(2) = 6$, so $\lim_{x\to 2} h(x) = h(2)$; Continuous on $(-\infty, \infty)$



g) Observe: $f(x)=\begin{cases}1, & \text{if } |x|>2\\ -1, & \text{if } |x|<2\end{cases}$

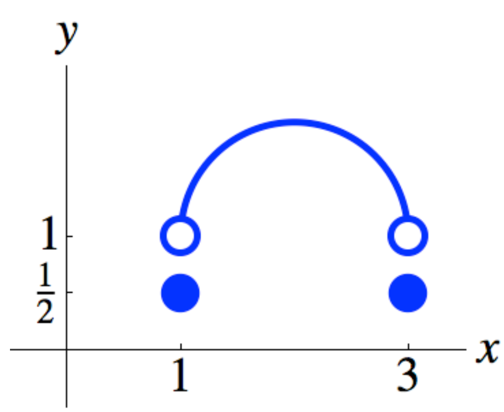
Discontinuities: -2 (jump: $\lim_{x\to -2^-} f(x)=1$, and $\lim_{x\to -2^+} f(x)=-1$, but $1\neq -1$), and 2 (jump: $\lim_{x\to 2^-} f(x)=-1$, and $\lim_{x\to 2^+} f(x)=1$, but $-1\neq 1$);

Continuous on $(-\infty,-2),(-2,2),(2,\infty)$;



2) Jump discontinuity

3) For example,



4) $A=10$

5) $[1,3),(3,4),(4,\infty)$

6) $\left\{x\in\mathbb{R}\,\middle|\, x\neq \dfrac{1}{\pi^2 n^2}\ \left(n\in\mathbb{Z}^+;\text{ i.e., } n \text{ is a positive integer}\right),\ \text{ and } x>0\right\}$

7) a) Let $f(x) = 3x^3 - 2x^2 - 2x - 5$. f is continuous on $[1, 2]$, so the IVT applies. $f(1) = -6$, and $f(2) = 7$. $0 \in [-6, 7]$, so, by the IVT, $\exists c \in [1, 2] \ni f(c) = 0$; such a value for c is a solution to the given equation.

b)

$$\begin{array}{r|rrrr} 5/3 & 3 & -2 & -2 & -5 \\ & & 5 & 5 & 5 \\ \hline & 3 & 3 & 3 & \vert\ 0 \end{array}$$

8) a) s is continuous on $[0, 1]$, so the IVT applies. $s(0) = 4$ (feet), and $s(1) = 18$ (feet). $15 \in [4, 18]$, so, by the IVT, $\exists c \in [0, 1] \ni s(c) = 15$. Such a value for c is a time (in seconds within one second after the projectile is fired) that the projectile achieves a height of 15 feet.

b) $t = \dfrac{1}{2}$ of a second

9) f is continuous on $\mathbb{R}$; in particular, it is continuous on $[1, 3]$, so the IVT applies.

$f(1) = 6$, $f(3) = 14$. Let $d \in [6, 14]$, and let $c = \sqrt{d - 5}$; observe:

$f(c) = d$ and $c \in [1, 3] \iff c^2 + 5 = d$ and $c \in [1, 3]$

$\iff c^2 = d - 5$ and $c \in [1, 3] \iff c = \sqrt{d - 5}$, a value in $[1, 3]$.

Observe: $6 \le d \le 14 \iff 1 \le d - 5 \le 9 \iff 1 \le \sqrt{d - 5} \le 3$.

Then, $c \in [1, 3]$, and $f(c) = c^2 + 5 = \left(\sqrt{d - 5}\right)^2 + 5 = (d - 5) + 5 = d$.

Therefore, $\forall d \in [6, 14], \exists c \in [1, 3] \ni f(c) = d$.

10) Hint 1: Use the Quadratic Formula. Hint 2: You will choose $c = \sqrt{d + 5} - 2$.

CHAPTER 3: DERIVATIVES

SECTION 3.1: DERIVATIVES, TANGENT LINES, and RATES OF CHANGE

In these Exercises, use a version of the Limit Definition of the Derivative.
Do **not** use the short cuts that will be introduced in later sections.

1) Let $f(x) = 5x^2 + 1$.

a) Evaluate the difference quotients in the tables below.

$\dfrac{f(3.1) - f(3)}{3.1 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = 0.1$	
$\dfrac{f(3.01) - f(3)}{3.01 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = 0.01$	
$\dfrac{f(3.001) - f(3)}{3.001 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = 0.001$	

$\dfrac{f(2.9) - f(3)}{2.9 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = -0.1$	
$\dfrac{f(2.99) - f(3)}{2.99 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = -0.01$	
$\dfrac{f(2.999) - f(3)}{2.999 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = -0.001$	

b) Does the table in a) rigorously prove what $f'(3)$ is?

c) Find $f'(3)$ rigorously using a version of the Limit Definition of the Derivative.

2) Let $f(x) = \sqrt{3x - 2}$. Consider the graph of $y = f(x)$ in the usual *xy*-plane.

   a) Find the slope of the tangent line to the graph of *f* at the point $(a, f(a))$, where *a* is an arbitrary real number in $\left(\frac{2}{3}, \infty\right)$.

   b) Find an equation of the **tangent line** to the graph of *f* at the point $(9, f(9))$.

   c) Find an equation of the **normal line** to the graph of *f* at the point $(9, f(9))$.

3) The position function *s* of a particle moving along a coordinate line is given by $s(t) = 2t - 3t^2$, where time *t* is measured in seconds, and $s(t)$ is measured in centimeters.

   a) Find the **average velocity** of the particle in the following time intervals:

      i. $[1, 1.1]$ ii. $[1, 1.01]$

   b) Find the **[instantaneous] velocity** of the particle at time $t = 1$.

SECTION 3.2: DERIVATIVE FUNCTIONS and DIFFERENTIABILITY

1) Let $f(x) = \frac{1}{x^2}$.

   a) Use the Limit Definition of the Derivative to find $f'(x)$.

   b) Use the short cuts to find $f'(x)$.

2) Let $r(x) = x^4$.

   a) Use the Limit Definition of the Derivative to find $r'(x)$.
   Hint: Use the Binomial Theorem from Section 9.5 in the Precalculus notes.

   b) Use the short cuts to find $r'(x)$.

3) Let $f(x) = 9\left(\sqrt[3]{x^2}\right)$. Find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.
   Do not leave negative exponents in your final answers.
   You do not have to simplify radicals or rationalize denominators.

4) Let $y = x^{10}$. What is $\frac{d^{20}y}{dx^{20}}$? You shouldn't have to show any work!

5) The position function s of a particle moving along a coordinate line is given by $s(t) = 4t^5$, where time t is measured in hours, and $s(t)$ is measured in miles.

   a) Determine the velocity function [rule] $v(t)$. (Use the short cuts.)

   b) Determine the velocity of the particle at times $t = 1$, $t = 2$, and $t = -4.7$.

   c) Determine the acceleration function [rule] $a(t)$. (Use the short cuts.)

   d) Determine the acceleration of the particle at times $t = 1$, $t = 2$, and $t = -4.7$.

6) For each part below, answer "Yes" or "No."

a) If $f(x) = x^4 - 3x + 1$, is f differentiable everywhere on $\mathbb{R}$?

b) If $g(x) = |3x - 8|$, is g differentiable at $\frac{8}{3}$?

c) If $h(t) = \sqrt[3]{t^2}$, is h differentiable at 0?

d) If $p(x) = \begin{cases} 4x+3, & \text{if } x \leq -1 \\ x^2 - 1, & \text{if } x > -1 \end{cases}$, is p differentiable at -1?

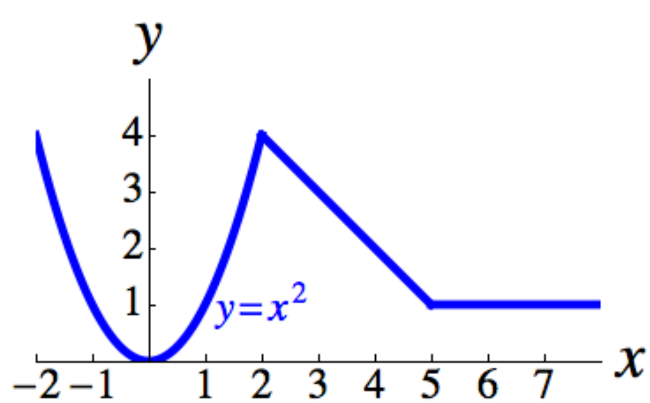
e) If $q(x) = \frac{3}{x+2}$, is q differentiable on the interval $(-10, 0)$?

f) If $q(x) = \frac{3}{x+2}$, is q differentiable on the interval $(0, 10)$?

7) Determine whether or not the graph of f has a **vertical tangent line** at the point $(0, 0)$ and whether f has a **corner**, a **cusp**, or **neither** at $(0, 0)$.

a) $f(x) = x^{4/5}$ b) $f(x) = x^{3/5}$ c) $f(x) = |x|$

8) The graph of f is given below. Sketch the graph of f'.



SECTION 3.3: TECHNIQUES OF DIFFERENTIATION

1) Let $g(w) = 3w^2 - 5w + 4$.

   a) Use the Limit Definition of the Derivative to find $g'(w)$.

   b) Use the short cuts to find $g'(w)$.

2) Find the following derivatives. Simplify where appropriate.
   Do not leave negative exponents in your final answer.
   You do not have to simplify radicals or rationalize denominators.
   From this point on, use the short cuts for differentiation, unless otherwise stated.

   a) Let $f(x) = 5x^3 - \dfrac{3}{x^2} + \dfrac{1}{\sqrt{x}} + \dfrac{\sqrt[3]{x}}{6} - 2$. Find $f'(x)$.

   b) Find $\dfrac{d}{dt}\left(\dfrac{3t-2}{5-t}\right)$.

   c) Find $D_z\left[\left(z^2-4\right)^2\right]$. Also, factor your result completely over the integers.

   d) Let $q = \left(w^2 - 3w + 1\right)\left(w^3 - 2\right)$. Find $\dfrac{dq}{dw}$.
      Use the Product Rule of Differentiation.
      After finding all relevant derivatives, you do not have to simplify further.

   e) Let $y = \dfrac{x^2 + 5x - 1}{2x^2}$. Find $\dfrac{dy}{dx}$; that is, find y'.
      i. Do this without using the Quotient Rule of Differentiation.
      ii. Do this using the Quotient Rule of Differentiation, and check that your answer is equivalent to your answer in i.

   f) Let $N(x) = \dfrac{\frac{4}{x^2}}{\frac{3}{x} + 2}$. Find $N'(x)$. Hint: Simplify first.

   g) Let $W(x) = (3x)^2$. Find $W'(x)$.

   h) Let $S(x) = (3x+1)^{-2}$. Find $S'(x)$. You will be able to find a shorter solution when you learn the Generalized Power Rule in Section 3.6.

3) The position function s of a particle moving along a coordinate line is given by $s(t) = 4t^3 + 15t^2 - 18t + 1$, where time t is measured in minutes, and $s(t)$ is measured in feet.

   a) Determine the velocity function [rule] $v(t)$. (Use the short cuts.)

   b) Determine the velocity of the particle at times $t = 1$, $t = 2$, and $t = -4.7$.

   c) Determine the acceleration function [rule] $a(t)$. (Use the short cuts.)

   d) Determine the acceleration of the particle at times $t = 1$, $t = 2$, and $t = -4.7$.

4) Consider the graph of $y = \dfrac{8}{x^2 + 4}$.

   a) Find an equation of the **tangent line** to the graph at the point $P(2, 1)$.

   b) Find an equation of the **normal line** to the graph at the point $P(2, 1)$.

5) (Product Rule for Products of Three Factors).
Assume that f, g, and h are functions that are everywhere differentiable on $\mathbb{R}$.
Use the Product Rule for Products of Two Factors given for this section to prove:

$$D_x\left[f(x)g(x)h(x)\right] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

6) (Preview of the Generalized Power Rule in Section 3.6).
Use Exercise 5 to find an expression for $D_x\left(\left[f(x)\right]^3\right)$.

7) Let $f(x) = x^3 - \dfrac{5}{2}x^2 - 2x + 1$, and let Point P be at $(1, f(1))$.

   a) Find the points on the graph of $y = f(x)$ at which the tangent line is horizontal.

   b) Find an equation of the **tangent line** to the graph of f at Point P.

   c) Find an equation of the **normal line** to the graph of f at Point P.

   d) Find the points on the graph of $y = f(x)$ at which the tangent line has a slope of 10.

8) A new military plane is flying over the ocean. The plane's flight path can be modeled by the graph of $y = x^2 + 100$, where x corresponds to horizontal position in feet, and y corresponds to the height of the plane in feet.

a) A target is placed at the point $P(0, 0)$. The plane is equipped with a double-tipped missile that can be shot in either direction tangent to the plane's path at the point the missile is shot. Find the two points along its path where the plane can shoot the missile in order to hit the target.

b) Repeat part a), except assume that the target is placed at the point $Q(24, 0)$.

Optional / Just for fun: Sketch the graph of $y = x^2 + 100$, and see why your answers to part b) differ the way they do from your answers to part a).

SECTION 3.4: DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

LIMITS (ADDITIONAL PROBLEMS)

1) Evaluate the following limits.

a) $\lim_{x \to 0} \dfrac{\sin x}{4x}$

b) $\lim_{x \to 0} \dfrac{\sin(5x)}{\sin(3x)}$

c) $\lim_{\theta \to 0} \dfrac{\tan^3(10\theta)}{\theta^3}$

d) $\lim_{x \to 0} \dfrac{1-\cos^3 x}{x}$

DERIVATIVES

2) Find the following derivatives. Simplify where appropriate.

a) Let $f(x) = x^5 \cos x$. Find $f'(x)$.

b) Let $g = \dfrac{\sin w}{1+\cos w}$. Find $\dfrac{dg}{dw}$.

c) Find $D_r(\csc r - \cot r)$.

d) Find $D_\alpha(7\sec\alpha + 4\alpha^2 - 2)$.

e) Find $D_\theta(\theta^2 \tan\theta)$.

f) Let $k(\beta) = \cos\beta \sec\beta$. Find $k'(\beta)$, and find the domain of k'.

TANGENT AND NORMAL LINES

3) Find the x-coordinates of all points on the graph of $y = 2\cos x + (\sqrt{2})x$ at which the tangent line is horizontal.

4) Consider the graph of $y = 1 + 2\cos x$.

   a) Find the x-coordinates of all points on the graph at which the tangent line is horizontal. (Suggestion: Sketch the graph, and see if your answer makes sense.)

   b) Find the x-coordinates of all points on the graph at which the tangent line is perpendicular to the line $y = -\frac{\sqrt{3}}{3}x + 4$.

   c) Find equations of the tangent line and the normal line to the graph at the point where the graph crosses the y-axis.

5) Find equations of the tangent line and the normal line to the graph of $y = \tan x - \sqrt{2}\sin x$ at the point $P\left(-\frac{3\pi}{4}, 2\right)$.

6) Find the x-coordinates of all points on the graph of $y = 2\sin x - \cos(2x)$ at which the tangent line is horizontal. Note: This problem will be revisited later when we discuss the Chain Rule.

PROOFS (SEE ALSO SECTION 3.6 EXERCISES)

In 7) and 8) below, do **not** use the Limit Definition of the Derivative.
You may use your knowledge of the derivatives of $\sin x$ and $\cos x$ without proof.

7) Prove: $D_x(\cot x) = -\csc^2 x$.

8) Prove: $D_x(\csc x) = -\csc x \cot x$.

KNOW THE FOLLOWING

- The derivatives of the six basic trig functions.

- The proofs for the derivatives of $\sin x$ and $\cos x$ using the Limit Definition of the Derivative.

- The proofs for the derivatives of the other four basic trig functions, assuming you have knowledge of the derivatives of $\sin x$ and $\cos x$.

SECTION 3.5: DIFFERENTIALS and LINEARIZATION OF FUNCTIONS

1) Use differentials and $\sqrt[4]{16}$ to approximate $\sqrt[4]{15.92}$.
   (Just for fun, compare your approximation to a calculator's result for $\sqrt[4]{15.92}$.)

2) Let $f(x) = -3x^3 + 8x - 7$. Find a linear approximation for $f(3.96)$ if x changes from 4 to 3.96.

3) Find a linear approximation for $\sec 31.5^\circ$ by using the value of $\sec 30^\circ$.
   Give your result as an exact value, and also give a decimal approximation rounded off to four significant digits.

SECTION 3.6: CHAIN RULE

1) Find the following derivatives. Simplify where appropriate.

a) Let $f(x) = \left(x^2 - 3x + 8\right)^3$. Find $f'(x)$.

b) Let $n = \dfrac{1}{\left(m^2 + 4\right)^5}$. Find $\dfrac{dn}{dm}$.

c) Let $h(x) = \left(x^2 - \dfrac{1}{x^2}\right)^6$. Find $h'(x)$.

d) Find $D_t\left[\tan^3(6t)\right]$.

e) Find $D_x\left[x^3 \sin^2(2x)\right]$.

f) Find $D_t\left[\sqrt[3]{8t^3 + 27}\right]$.

g) Find $D_\theta\left(\left[\pi - 6\theta^5 + \csc(5\theta)\right]^7\right)$

h) Find $D_w\left[\dfrac{1 + \sin(2w)}{\cos(2w)}\right]$. Instead of applying the Quotient Rule of Differentiation, rewrite the expression first for a quicker solution.

i) Redo h), but apply the Quotient Rule of Differentiation and simplify.

j) Find $D_\phi\left[\cos\left(\sqrt{\phi}\right) + \sqrt{\cos\phi}\right]$.

k) Find $D_x\left[(6x-7)^3\left(8x^2+9\right)^4\right]$, factor your result completely over the integers, and simplify.

l) Find $D_x\left[\dfrac{\left(x^2 - 3\right)^2}{\sqrt{x^2 + 5}}\right]$. Write your answer as a single simplified fraction.

Do not leave negative exponents in your final answer.
You do not have to simplify radicals or rationalize denominators.

2) Let $S(x)=(3x+1)^{-2}$. Find $S'(x)$. Compare your solution to your longer solution in Section 3.3, Exercise 2h.

3) Find $D_x\left(\left[f(x)\right]^3\right)$. Refer back to Section 3.3, Exercise 6.

4) Find the x-coordinates of all points on the graph of $y=2\sin x-\cos(2x)$ at which the tangent line is horizontal. (This exercise was introduced in Section 3.4, Exercise 6, but it should be easier now.)

5) Assume $y=f(u)$, $u=g(t)$, and $t=h(r)$, where f, g, and h are functions that are all differentiable everywhere on $\mathbb{R}$. Given that $\frac{dy}{du}=2$, $\frac{du}{dt}=7$, and $\frac{dt}{dr}=3$, find $\frac{dy}{dr}$.

6) We will find $D_x\left[\left(x^3\right)^5\right]$ in three different ways. Observe that all three results are equivalent.

   a) Simplify $\left(x^3\right)^5$, and use the Basic Power Rule of Differentiation.

   b) Do not simplify $\left(x^3\right)^5$, use the Generalized Power Rule of Differentiation, and simplify.

   c) (Seeing the Chain Rule in action). Let $u=x^3$, let $y=u^5$, use the Chain Rule to find $\frac{dy}{dx}$, and simplify. Write your answer in terms of x alone, not x and u.

7) We will find $D_x\left(\dfrac{1}{x^2+1}\right)$ in three different ways. Observe that all three results are equivalent.

   a) Use the Reciprocal Rule or the Quotient Rule of Differentiation, and simplify.

   b) Rewrite $D_x\left(\dfrac{1}{x^2+1}\right)$ as $D_x\left[\left(x^2+1\right)^{-1}\right]$, use the Generalized Power Rule of Differentiation, and simplify.

   c) (Seeing the Chain Rule in action). Let $u = x^2 + 1$, let $y = \dfrac{1}{u}$, use the Chain Rule to find $\dfrac{dy}{dx}$, and simplify. Write your answer in terms of x alone, not x and u.

8) The graph of $y = \sqrt{a^2 - x^2}$, where $a > 0$, is the upper half of a **circle** of radius a centered at O (the origin). Show that the **tangent line** to any point P on the graph is **perpendicular** to the line segment $\overline{OP}$. (We will revisit this in Section 3.7, Exercise 4. This issue was brought up in Section 3.4, Part E.)

9) We can use the Cofunction Identities to prove that $D_x(\csc x) = -\csc x \cot x$.

   Complete the proof: $D_x(\csc x) = D_x\left[\sec\left(\dfrac{\pi}{2} - x\right)\right] = \ldots = -\csc x \cot x$.

   (You may use the Generalized Trigonometric Rule for the secant function.) This helps explain the patterns we find between pairs of derivative rules for cofunctions.

10)

   a) Find $D_x\left[\sin(2x)\right]$.

   b) If we didn't have the Chain Rule, we may have to resort to the Double-Angle Identities. Find $D_x\left[\sin(2x)\right]$ using the Double-Angle Identities.

   c) How does $D_x\left[\sin(2x)\right]$ compare with $D_x(\sin x)$? What does this tell us about how the graph of $y = \sin(2x)$ differs from the graph of $y = \sin x$?

SECTION 3.7: IMPLICIT DIFFERENTIATION

1) Consider the given equation $xy^2 + y = 4$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.

   a) Find $\frac{dy}{dx}$, also known as y'.

   b) Verify that the point $P\left(\frac{1}{2}, 2\right)$ lies on the graph of the given equation.

   c) Evaluate $\left[\frac{dy}{dx}\right]_{\left(\frac{1}{2}, 2\right)}$.

   d) Find an equation of the tangent line to the graph of the given equation at the point $P\left(\frac{1}{2}, 2\right)$.

2) Consider the given equation $5x^2 - 2xy + 3x^3y^4 - 4y^2 = 44$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.

   a) Find $\frac{dy}{dx}$, also known as y'.

   b) Verify that the point $P(2, -1)$ lies on the graph of the given equation.

   c) Evaluate $\left[\frac{dy}{dx}\right]_{(2, -1)}$.

   d) Find an equation of the tangent line to the graph of the given equation at the point $P(2, -1)$.

3) Consider the given equation $\sin\left(\sqrt{y}\right) + x\cos y = 3$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.

   a) Find $\frac{dy}{dx}$, also known as y'.

   b) Verify that the point $P(3, 0)$ lies on the graph of the given equation.

   c) Evaluate your result from a) at the point $P(3, 0)$. It turns out that P is an endpoint of the graph of the given equation, and your result here corresponds to a kind of one-sided derivative.

4) Use Implicit Differentiation to show that the **tangent line** to any point P on a **circle** with center O (the origin) is **perpendicular** to the line segment $\overline{OP}$. (This is another approach to Section 3.6, Exercise 8. This issue was brought up in Section 3.4, Part E.)

SECTION 3.8: RELATED RATES

1) Assuming $3x^2y + 2x = -32$, where x and y are differentiable functions of t, and $\frac{dy}{dt} = -4$ when $x = 2$ and $y = -3$, find $\frac{dx}{dt}$.

2) (Balloon problem). Air is being pumped into a spherical balloon at the rate of 1.754 cubic centimeters per second. The balloon maintains a spherical shape throughout. How fast is the radius of the balloon changing when the diameter is 2.736 centimeters in length? Round off your answer to four significant digits. Keep intermediate results exact.

3) (Ladder problem). A wall stands upright and perpendicular from the flat ground. A 25-foot long ladder leans against the wall. The bottom of the ladder is moved away from the building horizontally (along a line perpendicular to the wall) at a rate of 30 inches per minute (until the ladder lies flat on the ground). How fast is the top of the ladder sliding down the building when the top of the ladder is 10 feet above the ground? Round off your answer to four significant digits.
   Keep intermediate results exact.

4) (Two bug problem). A ladybug crawls out of a small hole in a large wall and crawls to the right at a rate of 3 inches per minute. Forty-five seconds later, a tick crawls out of the hole and crawls up at a rate of 2 inches per minute. How fast is the distance between the ladybug and the tick changing four minutes after the ladybug crawls out of the hole? Round off your answer to three significant digits. Keep intermediate results exact.

5) (Boyle's Law for Ideal Gases). If the temperature and the mass of a confined ideal gas are fixed, then $PV = k$, where P is the pressure and V is the volume of the gas, and k is a constant. How fast is the volume of the gas changing at the moment that the pressure is $45\,\dfrac{\text{lb}}{\text{in}^2}$, the volume is 60 in^3, and the pressure is increasing at a rate of $3\,\dfrac{\text{lb}}{\text{in}^2}$ per minute? Give an exact answer.

6) (Cylinder problem). A right circular cylinder's volume is shrinking at the rate of $15\,\dfrac{\text{cm}^3}{\text{hr}}$ in such a way that its base radius is always twice its height. The cylinder retains a right circular cylindrical shape. How fast is the base radius changing when the height is 1 meter? Give an exact answer.

7) (Parallel resistor problem). Two parallel resistors have resistances R_1 and R_2. If the total resistance is R, then $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. R_1 is increasing at $0.04\,\dfrac{\text{ohm}}{\text{sec}}$, and R_2 is decreasing at $0.03\,\dfrac{\text{ohm}}{\text{sec}}$. How fast is the total resistance changing at the moment that R_1 is 3 ohms and R_2 is 4 ohms? Round off your answer to four significant digits.

8) (Airplane problem). A military plane maintains an altitude of 15,000 feet over a vast flat desert. It flies at a constant speed on a line that will take it directly over an observer on the ground. At noon, the angle of elevation from the observer's shoes to the plane is 30 degrees, and the angle of elevation is increasing at a rate of 2.2 degrees per second. Find the speed of the plane at noon. Round off your answer to five significant digits. What is the speed in miles per hour (mph)? Remember that there are 5280 feet in one mile.

CHAPTER 3: DERIVATIVES

SECTION 3.1: DERIVATIVES, TANGENT LINES, and RATES OF CHANGE

1) a)

$\frac{f(3.1)-f(3)}{3.1-3}$; this is $\frac{f(3+h)-f(3)}{h}$ with $h=0.1$	30.5
$\frac{f(3.01)-f(3)}{3.01-3}$; this is $\frac{f(3+h)-f(3)}{h}$ with $h=0.01$	30.05
$\frac{f(3.001)-f(3)}{3.001-3}$; this is $\frac{f(3+h)-f(3)}{h}$ with $h=0.001$	30.005

$\frac{f(2.9)-f(3)}{2.9-3}$; this is $\frac{f(3+h)-f(3)}{h}$ with $h=-0.1$	29.5
$\frac{f(2.99)-f(3)}{2.99-3}$; this is $\frac{f(3+h)-f(3)}{h}$ with $h=-0.01$	29.95
$\frac{f(2.999)-f(3)}{2.999-3}$; this is $\frac{f(3+h)-f(3)}{h}$ with $h=-0.001$	29.995

b) No

c) 30

2) a) $f'(a)=\frac{3}{2\sqrt{3a-2}}$. Hint: Rationalize the numerator of the difference quotient.

b) Point-Slope Form: $y-5=\frac{3}{10}(x-9)$, Slope-Intercept Form: $y=\frac{3}{10}x+\frac{23}{10}$

c) Point-Slope Form: $y-5=-\frac{10}{3}(x-9)$, Slope-Intercept Form: $y=-\frac{10}{3}x+35$

3) a) i. $-4.3\frac{\text{cm}}{\text{sec}}$, ii. $-4.03\frac{\text{cm}}{\text{sec}}$; b) $-4\frac{\text{cm}}{\text{sec}}$

SECTION 3.2: DERIVATIVE FUNCTIONS and DIFFERENTIABILITY

1) Hint: $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$; $f'(x) = -\dfrac{2}{x^3}$

2) Hint: $\dfrac{r(x+h)-r(x)}{h} = \dfrac{\left(x^4+4x^3h+6x^2h^2+4xh^3+h^4\right)-x^4}{h}$; $r'(x)=4x^3$

3) $f'(x)=\dfrac{6}{x^{1/3}}$, or $\dfrac{6}{\sqrt[3]{x}}$, $f''(x)=-\dfrac{2}{x^{4/3}}$, or $-\dfrac{2}{\sqrt[3]{x^4}}$, $f'''(x)=\dfrac{8}{3x^{7/3}}$, or $\dfrac{8}{3\left(\sqrt[3]{x^7}\right)}$,

$f^{(4)}(x)=-\dfrac{56}{9x^{10/3}}$, or $-\dfrac{56}{9\left(\sqrt[3]{x^{10}}\right)}$

4) 0

5) a) $v(t)=20t^4$

b) $v(1)=20$ mph, $v(2)=320$ mph, $v(-4.7)=9759.362$ mph; mph = miles per hour

c) $a(t)=80t^3$

d) $a(1)=80\,\dfrac{\text{mi}}{\text{hr}^2}$, $a(2)=640\,\dfrac{\text{mi}}{\text{hr}^2}$, $a(-4.7)=-8305.84\,\dfrac{\text{mi}}{\text{hr}^2}$

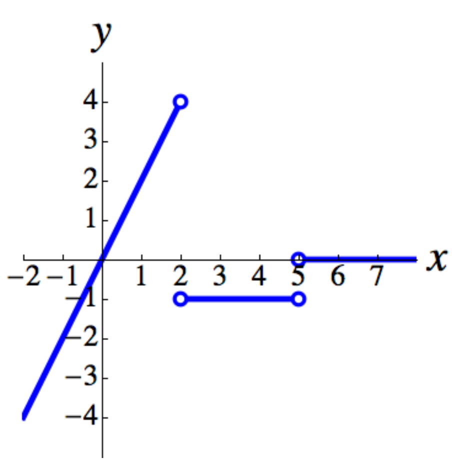
6) a) Yes; b) No; c) No; d) No (observe that p is discontinuous at -1); e) No; f) Yes

7) a) Yes, there is a vertical tangent line; a cusp

b) Yes, there is a vertical tangent line; neither a corner nor a cusp

c) No, there is not a vertical tangent line; a corner

8)



SECTION 3.3: TECHNIQUES OF DIFFERENTIATION

1) Hint: $\dfrac{g(w+h)-g(w)}{h}=\dfrac{\left[3(w+h)^2-5(w+h)+4\right]-\left[3w^2-5w+4\right]}{h}$;

$g'(w)=6w-5$

2) a) $15x^2+\dfrac{6}{x^3}-\dfrac{1}{2x^{3/2}}+\dfrac{1}{18x^{2/3}}$, or $15x^2+\dfrac{6}{x^3}-\dfrac{1}{2\sqrt{x^3}}+\dfrac{1}{18\left(\sqrt[3]{x^2}\right)}$

b) $\dfrac{13}{(5-t)^2}$, or $\dfrac{13}{(t-5)^2}$

c) $4z^3-16z$; this can be factored as $4z(z+2)(z-2)$.

d) $(2w-3)(w^3-2)+(w^2-3w+1)(3w^2)$

e) i. $\dfrac{1}{x^3}-\dfrac{5}{2x^2}$ (Hint: First, reexpress using algebra.),

ii. $\dfrac{2-5x}{2x^3}$, which is equivalent to $\dfrac{1}{x^3}-\dfrac{5}{2x^2}$.

f) $-\dfrac{4(3+4x)}{(3x+2x^2)^2}$, which could be "simplified" to $-\dfrac{4(3+4x)}{x^2(3+2x)^2}$; ask your instructor if s/he has a preference.

g) $18x$

h) $-\dfrac{6}{(3x+1)^3}$

3) a) $v(t)=12t^2+30t-18$

b) $v(1)=24\,\dfrac{\text{ft}}{\text{min}}$, $v(2)=90\,\dfrac{\text{ft}}{\text{min}}$, $v(-4.7)=106.08\,\dfrac{\text{ft}}{\text{min}}$

c) $a(t)=24t+30$

d) $a(1)=54\,\dfrac{\text{ft}}{\text{min}^2}$, $a(2)=78\,\dfrac{\text{ft}}{\text{min}^2}$, $a(-4.7)=-82.8\,\dfrac{\text{ft}}{\text{min}^2}$

4) a) Point-Slope Form: $y-1=-\frac{1}{2}(x-2)$, Slope-Intercept Form: $y=-\frac{1}{2}x+2$

b) Point-Slope Form: $y-1=2(x-2)$, Slope-Intercept Form: $y=2x-3$

5) Hint: $D_x\left[f(x)g(x)h(x)\right]=D_x\left(\left[f(x)g(x)\right]h(x)\right)$.

6) $3\left[f(x)\right]^2 f'(x)$. Hint: Assume that f, g, and h are equivalent functions.

7) a) $\left(-\frac{1}{3},\frac{73}{54}\right)$ and $(2,-5)$

b) Point-Slope Form: $y-\left(-\frac{5}{2}\right)=-4(x-1)$, Slope-Intercept Form: $y=-4x+\frac{3}{2}$

c) Point-Slope Form: $y-\left(-\frac{5}{2}\right)=\frac{1}{4}(x-1)$, Slope-Intercept Form: $y=\frac{1}{4}x-\frac{11}{4}$

d) $\left(3,-\frac{1}{2}\right)$ and $\left(-\frac{4}{3},-\frac{85}{27}\right)$

8) a) $(10,200)$ and $(-10,200)$. Hint: Find the point(s) $(a,f(a))$ on the flight path where the slope of the tangent line there equals the slope of the line connecting the point and the target.

b) $(-2,104)$ and $(50,2600)$

SECTION 3.4: DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1) a) $\frac{1}{4}$

b) $\frac{5}{3}$. Hint: $\lim_{x\to 0} \frac{\sin(5x)}{5x} = 1$.

c) 1000

d) 0. Hint: Factor the numerator.

2) a) $5x^4 \cos x - x^5 \sin x$, or $x^4(5\cos x - x\sin x)$

b) $\frac{1}{1+\cos w}$

c) $\csc^2 r - \csc r \cot r$, or $(\csc r)(\csc r - \cot r)$

d) $7\sec\alpha \tan\alpha + 8\alpha$

e) $2\theta \tan\theta + \theta^2 \sec^2\theta$, or $\theta(2\tan\theta + \theta \sec^2\theta)$

f) 0, $\mathrm{Dom}(k') = \left\{ \beta \in \mathbb{R} \,\middle|\, \beta \neq \frac{\pi}{2} + \pi n \quad (n \in \mathbb{Z}) \right\}$

3) $\left\{ x \in \mathbb{R} \,\middle|\, x = \frac{\pi}{4} + 2\pi n, \text{ or } x = \frac{3\pi}{4} + 2\pi n \quad (n \in \mathbb{Z}) \right\}$

4) a) $\left\{ x \in \mathbb{R} \,\middle|\, x = \pi n \quad (n \in \mathbb{Z}) \right\}$

b) $\left\{ x \in \mathbb{R} \,\middle|\, x = \frac{4\pi}{3} + 2\pi n, \text{ or } x = \frac{5\pi}{3} + 2\pi n \quad (n \in \mathbb{Z}) \right\}$

c) Tangent line: $y = 3$, Normal line: $x = 0$

5) Tangent line:

Point-Slope Form: $y-2=3\left(x-\left(-\frac{3\pi}{4}\right)\right)$,

Slope-Intercept Form: $y=3x+\frac{9\pi+8}{4}$;

Normal line:

Point-Slope Form: $y-2=-\frac{1}{3}\left(x-\left(-\frac{3\pi}{4}\right)\right)$,

Slope-Intercept Form: $y=-\frac{1}{3}x+\frac{8-\pi}{4}$

6) Most efficiently: $\left\{x\in\mathbb{R}\,\middle|\, x=\frac{\pi}{2}+\frac{2\pi}{3}n, \text{ or } x=-\frac{\pi}{2}+2\pi n \quad (n\in\mathbb{Z})\right\}$.

Equivalently, $\left\{x\in\mathbb{R}\,\middle|\, x=\frac{\pi}{2}+\pi n, \text{ or } x=-\frac{\pi}{6}+2\pi n, \text{ or } x=\frac{7\pi}{6}+2\pi n \ (n\in\mathbb{Z})\right\}$.

Hint 1: Use a Double-Angle ID.
Hint 2: $\sin^2 x=(\sin x)(\sin x)$.

7) Hint: $D_x(\cot x)=D_x\left(\frac{\cos x}{\sin x}\right)$

8) Hint: $D_x(\csc x)=D_x\left(\frac{1}{\sin x}\right)$

SECTION 3.5: DIFFERENTIALS and LINEARIZATION OF FUNCTIONS

1) 1.9975

2) -161.56

3) $\frac{2\sqrt{3}}{3}+\frac{\pi}{180}=\frac{120\sqrt{3}+\pi}{180}\approx 1.172$

SECTION 3.6: CHAIN RULE

1) a) $3\left(x^2-3x+8\right)^2\left(2x-3\right)$, or $3\left(2x-3\right)\left(x^2-3x+8\right)^2$

b) $-\dfrac{10m}{\left(m^2+4\right)^6}$

c) $6\left(x^2-\dfrac{1}{x^2}\right)^5\left(2x+\dfrac{2}{x^3}\right)$, or $12\left(x+\dfrac{1}{x^3}\right)\left(x^2-\dfrac{1}{x^2}\right)^5$, or $12\left(\dfrac{x^4+1}{x^3}\right)\left(\dfrac{x^4-1}{x^2}\right)^5$, or

$\dfrac{12\left(x^4+1\right)\left(x^4-1\right)^5}{x^{13}}$

d) $18\tan^2\left(6t\right)\sec^2\left(6t\right)$

e) $3x^2\sin^2\left(2x\right)+4x^3\sin\left(2x\right)\cos\left(2x\right)$, or $\left[x^2\sin\left(2x\right)\right]\left[3\sin\left(2x\right)+4x\cos\left(2x\right)\right]$, or

$3x^2\sin^2\left(2x\right)+2x^3\sin\left(4x\right)$, or $x^2\left[3\sin^2\left(2x\right)+2x\sin\left(4x\right)\right]$

f) $\dfrac{8t^2}{\left(8t^3+27\right)^{2/3}}$, or $\dfrac{8t^2}{\sqrt[3]{\left(8t^3+27\right)^2}}$, or $\dfrac{8t^2\left(\sqrt[3]{8t^3+27}\right)}{8t^3+27}$

g) $7\left[\pi-6\theta^5+\csc\left(5\theta\right)\right]^6\left[-30\theta^4-5\csc\left(5\theta\right)\cot\left(5\theta\right)\right]$, or

$-35\left[\pi-6\theta^5+\csc\left(5\theta\right)\right]^6\left[6\theta^4+\csc\left(5\theta\right)\cot\left(5\theta\right)\right]$

h) $2\sec\left(2w\right)\tan\left(2w\right)+2\sec^2\left(2w\right)$, or $\left[2\sec\left(2w\right)\right]\left[\tan\left(2w\right)+\sec\left(2w\right)\right]$

i) Same as h).

j) $-\dfrac{\sin\left(\sqrt{\phi}\right)}{2\sqrt{\phi}}-\dfrac{\sin\phi}{2\sqrt{\cos\phi}}$, or $-\dfrac{1}{2}\left[\dfrac{\sin\left(\sqrt{\phi}\right)}{\sqrt{\phi}}+\left(\sin\phi\right)\sqrt{\sec\phi}\right]$, or

$-\dfrac{\sqrt{\phi}}{2}\left[\dfrac{\sin\left(\sqrt{\phi}\right)+\left(\sin\phi\right)\sqrt{\phi\sec\phi}}{\phi}\right]$

k) $18(6x-7)^2(8x^2+9)^4+64x(6x-7)^3(8x^2+9)^3$, which factors and simplifies as $2(264x^2-224x+81)(6x-7)^2(8x^2+9)^3$. By the Test for Factorability from Section 0.7 in the Precalculus notes, the discriminant of $(264x^2-224x+81)$ is not a perfect square, so it cannot be factored further over the integers.

l) $\dfrac{x(x^2-3)(3x^2+23)}{(x^2+5)^{3/2}}$, or $\dfrac{x(x^2-3)(3x^2+23)}{\sqrt{(x^2+5)^3}}$, or $\dfrac{x(x^2-3)(3x^2+23)}{(x^2+5)\sqrt{x^2+5}}$, or $\dfrac{x(x^2-3)(3x^2+23)\sqrt{x^2+5}}{(x^2+5)^2}$

2) $-\dfrac{6}{(3x+1)^3}$

3) $3[f(x)]^2 f'(x)$, just like in Section 3.3, Exercise 4.

4) Most efficiently: $\left\{ x \in \mathbb{R} \,\middle|\, x = \dfrac{\pi}{2} + \dfrac{2\pi}{3}n, \text{ or } x = -\dfrac{\pi}{2} + 2\pi n \quad (n \in \mathbb{Z}) \right\}$.

Equivalently, $\left\{ x \in \mathbb{R} \,\middle|\, x = \dfrac{\pi}{2} + \pi n, \text{ or } x = -\dfrac{\pi}{6} + 2\pi n, \text{ or } x = \dfrac{7\pi}{6} + 2\pi n \ (n \in \mathbb{Z}) \right\}$,

just like in Section 3.4, Exercise 6.

5) 42

6) a) $D_x\left[\left(x^3\right)^5\right] = D_x\left[x^{15}\right] = 15x^{14}$

b) $D_x\left[\left(x^3\right)^5\right] = 5\left(x^3\right)^4\left(3x^2\right) = 15x^{14}$

c) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(5u^4\right)\left(3x^2\right) = 15x^2u^4 = 15x^2\left(x^3\right)^4 = 15x^{14}$

7) a) $D_x\left(\frac{1}{x^2+1}\right) = -\frac{D_x\left(x^2+1\right)}{\left(x^2+1\right)^2} = -\frac{2x}{\left(x^2+1\right)^2}$

b) $D_x\left(\frac{1}{x^2+1}\right) = D_x\left[\left(x^2+1\right)^{-1}\right] = -\left(x^2+1\right)^{-2}(2x) = -\frac{2x}{\left(x^2+1\right)^2}$

c) $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(-\frac{1}{u^2}\right)(2x) = -\frac{2x}{u^2} = -\frac{2x}{\left(x^2+1\right)^2}$

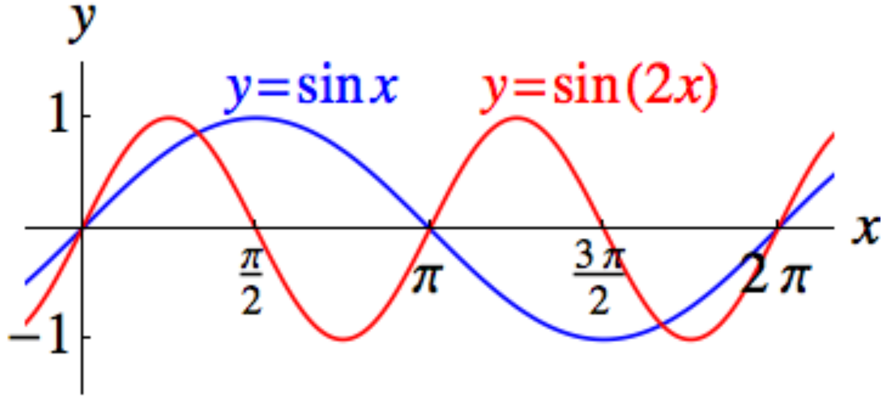
8) Hints: How are slopes of perpendicular lines (or line segments) related? $D_x\left(\sqrt{a^2-x^2}\right) = -\frac{x}{\sqrt{a^2-x^2}}$. Horizontal and vertical tangent lines correspond to special cases.

9) Hint: You will need the Cofunction Identities again: $\sec\left(\frac{\pi}{2}-x\right) = \csc x$ and $\tan\left(\frac{\pi}{2}-x\right) = \cot x$.

10) a) $2\cos(2x)$

b) $D_x\left[\sin(2x)\right] = D_x\left[2\sin x\cos x\right] = \ldots = 2\left(\cos^2 x - \sin^2 x\right) = 2\cos(2x)$

c) The range of $D_x\left[\sin(2x)\right]$ is $[-2,2]$. The range of $D_x(\sin x)$ is $[-1,1]$. This tells us, among other things, that the steepest tangent lines to the graph of $y=\sin(2x)$ are twice as steep as the steepest tangent lines to the graph of $y=\sin x$. More incisively, the slope of the tangent line to the graph of $y=\sin(2x)$ at $x=a$ is twice the slope of the tangent line to the graph of $y=\sin x$ at $x=2a$, where a is any real value.



SECTION 3.7: IMPLICIT DIFFERENTIATION

1) a) $\dfrac{dy}{dx} = -\dfrac{y^2}{2xy+1}$

b) $\left(\dfrac{1}{2}\right)(2)^2 + (2) = 4$

c) $-\dfrac{4}{3}$

d) Point-Slope Form: $y - 2 = -\dfrac{4}{3}\left(x - \dfrac{1}{2}\right)$, Slope-Intercept Form: $y = -\dfrac{4}{3}x + \dfrac{8}{3}$

2) a) $\dfrac{dy}{dx} = \dfrac{2y - 10x - 9x^2y^4}{2\left(6x^3y^3 - x - 4y\right)}$

b) $5(2)^2 - 2(2)(-1) + 3(2)^3(-1)^4 - 4(-1)^2 = 44$

c) $\dfrac{29}{46}$

d) Point-Slope Form: $y - (-1) = \dfrac{29}{46}(x - 2)$, Slope-Intercept Form: $y = \dfrac{29}{46}x - \dfrac{52}{23}$

3) a) $\dfrac{dy}{dx} = \dfrac{2\sqrt{y}\cos y}{2x\sqrt{y}\sin y - \cos\left(\sqrt{y}\right)}$

b) $\sin\left(\sqrt{0}\right) + 3\cos 0 = 3$

c) 0

4) Hints: Consider the equation $x^2 + y^2 = a^2$, where $a > 0$. How are slopes of perpendicular lines (or line segments) related? Horizontal and vertical tangent lines correspond to special cases.

SECTION 3.8: RELATED RATES

1) $-\dfrac{24}{17}$

2) The radius is increasing at about $0.07458\ \dfrac{\text{cm}}{\text{sec}}$. Note: $\dfrac{1.754}{4\pi(1.368)^2} \approx 0.07458$.

3) The top of the ladder is sliding down at about $5.728\ \dfrac{\text{ft}}{\text{min}}$. Exact: $\dfrac{5\sqrt{21}}{4}\ \dfrac{\text{ft}}{\text{min}}$.

4) The distance is increasing at about $3.59\ \dfrac{\text{in}}{\text{min}}$. Exact: $\dfrac{98\sqrt{745}}{745}\ \dfrac{\text{in}}{\text{min}}$.

5) The volume is decreasing at $4\ \dfrac{\text{in}^3}{\text{min}}$.

6) The base radius is shrinking at $\dfrac{1}{4000\pi}\ \dfrac{\text{cm}}{\text{hr}}$.

7) The total resistance is increasing at about $0.007551\ \dfrac{\text{ohm}}{\text{sec}}$. Exact: $\dfrac{37}{4900}\ \dfrac{\text{ohm}}{\text{sec}}$.

Hint: From the given equation, $R = \dfrac{12}{7}$ ohms at that moment.

8) The plane's speed is about $2303.8\ \dfrac{\text{ft}}{\text{sec}}$, or about 1570.8 mph.

Exact: $\dfrac{2200\pi}{3}\ \dfrac{\text{ft}}{\text{sec}}$, or 500π mph.

CHAPTER 4: APPLICATIONS OF DERIVATIVES

SECTION 4.1: EXTREMA

1) For each part below, find the absolute maximum and minimum values of f on the given interval. Also give the absolute maximum and minimum points on the graph of $y = f(x)$. Show work!

   a) $f(x) = 15 + 8x - 2x^2$ on $[-1, 3]$

   b) $f(x) = \frac{1}{4}x^4 - \frac{5}{3}x^3 - 3x^2 + 10$ on $[-1, 2]$

   c) $f(x) = x^2 - \frac{16}{x}$ on $[-4, -1]$

2) For each part below, find the domain and the critical number(s) ("CNs") of the function with the indicated rule. If there are no critical numbers, write "NONE."

   a) $f(x) = 4x^4 - x^3 - 32x^2 + 12x + 13$

   b) $g(t) = \sqrt{t^2 - 36}$

   c) $h(x) = x\sqrt{4x - 1}$

   d) $p(\theta) = \frac{1}{2}\cos(2\theta) + \sin\theta$

   e) $q(x) = \cot x$

SECTION 4.2: MEAN VALUE THEOREM (MVT) FOR DERIVATIVES

1) For each part below, determine whether or not f satisfies the hypotheses of Rolle's Theorem on the given interval $[a, b]$. If it does not, explain why not. If it does, find all real values c in (a, b) that satisfy the conclusion of the theorem; i.e., $f'(c) = 0$.

    a) $f(x) = x^2 - 6x + 10$ on $[1, 5]$

    b) $f(x) = x^2 - 6x + 10$ on $[3, 7]$

    c) $f(x) = x^4 + 14x^3 + 69x^2 + 140x - 5$ on $[-6, -1]$;

    Hint: -2 is a value for c that satisfies $f'(c) = 0$.

    d) $f(x) = |x|$ on $[-4, 4]$

2) For each part below, determine whether or not f satisfies the hypotheses of the Mean Value Theorem (MVT) for Derivatives on the given interval $[a, b]$. If it does not, explain why not. If it does, find all real values c in (a, b) that satisfy the conclusion of the theorem; i.e., $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

    a) $f(x) = x + \dfrac{4}{x}$ on $[1, 4]$

    b) $f(x) = 2x^3 + 5x^2 - 4x + 3$ on $[-2, 3]$

    c) $f(x) = x^{2/3}$ on $[-8, 8]$

    d) $f(x) = 3x + 1$ on $[0, 2]$

KNOW THE FOLLOWING

- Rolle's Theorem
- The Mean Value Theorem (MVT) for Derivatives

SECTION 4.3: FIRST DERIVATIVE TEST

1) For each part below, sketch the graph of $y = f(x)$.

   - Find the domain of *f*.
   - State whether f is even, odd, or neither, and incorporate any corresponding symmetry in your graph.
   - Find the *y*-intercept, if any. You do not have to find *x*-intercepts.
   - Find and indicate on your graph any holes, vertical asymptotes (VAs), horizontal asymptotes (HAs), and slant asymptotes (SAs), and justify them using limits.
   - Find all points at critical numbers (if any). Indicate these points on your graph.
   - Use the First Derivative Test to classify each point at a critical number as a local maximum point, a local minimum point, or neither. (The next instruction may help.)
   - Find the intervals on which f is increasing / decreasing, and have your graph show that.

   a) $f(x) = -x^3 + 3x$. You can find the *x*-intercepts here.

   b) $f(x) = x^4 + 14x^3 + 69x^2 + 140x - 5$.
   Hint 1: You studied this in Section 4.2, Exercise 1c.
   Hint 2: The Remainder Theorem from Section 2.3 in the Precalculus notes may help you with function evaluations.
   Optional / For fun: See how the graph relates to Section 4.2, Exercise 1c.

   c) $f(x) = \dfrac{1}{x-4}$

   d) $f(x) = \sin x - \dfrac{1}{2}x$; restrict the domain to $(-2\pi, 2\pi)$. Symmetry will help!

2) f is a function that is continuous everywhere on $\mathbb{R}$ such that 7 is the only critical number, $f'(\pi) = 11$, and $f'(13) = -2$.

   a) According to the First Derivative Test, is the point $(7, f(7))$ on the graph of $y = f(x)$ a local maximum point, a local minimum point, or neither?

   b) Can we conclude from the given information that f is everywhere differentiable on $\mathbb{R}$, except possibly at 7? Why or why not?

SECTION 4.4: SECOND DERIVATIVES

1) Let's revisit Section 4.3, Exercise 1. For each part below …
   - Find the Possible Inflection Numbers (PINs), if any.
   - Find the x-intervals on which the graph of $y = f(x)$ is concave up / concave down, and see how this is consistent with your graph in Section 4.3.
   - For each PIN, state whether or not the corresponding point on the graph is an inflection point (IP). Find the y-coordinate(s) of the point(s) in a) and d) but not b) and c).

   a) $f(x) = -x^3 + 3x$

   b) $f(x) = x^4 + 14x^3 + 69x^2 + 140x - 5$

   c) $f(x) = \dfrac{1}{x-4}$

   d) $f(x) = \sin x - \dfrac{1}{2}x$; restrict the domain to $(-2\pi, 2\pi)$.
      Find the inflection points (IPs).

2) Let $f(x) = x^4 + 14x^3 + 69x^2 + 140x - 5$, as in Exercise 1b.
   In Section 4.3, Exercise 1b, you should have used the First Derivative Test to show that f has a local minimum at $x = -5$. Prove this using the Second Derivative Test, instead.

3) Let $g(\theta) = 4\cos^2(3\theta)$. Hint: A trig ID will prove very helpful here.

   a) What does the Second Derivative Test tell us about the point on the graph of $y = g(\theta)$ where $\theta = \dfrac{\pi}{3}$? Justify your answer.

   b) What does the Second Derivative Test tell us about the point on the graph of $y = g(\theta)$ where $\theta = \dfrac{\pi}{4}$? Justify your answer.

4) Let $h(t) = t^6 + 3t^4$. What does the Second Derivative Test tell us about the point $(0, 0)$ on the graph of $y = h(t)$? Justify your answer.

5) In 2009, supporters of President Obama said that, when it came to employment during his first year in office, the first derivative was negative, but the second derivative was positive. Interpret this statement.

SECTION 4.5: GRAPHING

1) For each part below, sketch the graph of $y = f(x)$.

- Find the domain of f.
- State whether f is even, odd, or neither, and incorporate any corresponding symmetry in your graph.
- Find the y-intercept, if any. You do not have to find x-intercepts in a), but find any x-intercepts in the others.
- Find and indicate on your graph any holes, vertical asymptotes (VAs), horizontal asymptotes (HAs), and slant asymptotes (SAs), and justify them using limits.
- Find the critical numbers (CNs), if any.
- Find all points at critical numbers (if any). Indicate these points on your graph.
- Use the First Derivative Test to classify each point at a critical number as a local maximum point, a local minimum point, or neither. (The next instruction may help.)
- Find the intervals on which f is increasing / decreasing, and have your graph show that.
- Find the "Possible Inflection Numbers" (PINs), if any.
- Find the x-intervals on which the graph of $y = f(x)$ is concave up / concave down, and have your graph show that.
- For each PIN, state whether or not the corresponding point on the graph is an inflection point (IP). Find any IPs.

a) $f(x) = -x^4 + 4x^3 + 48x^2 + 112x - 500$.
Hint: -2 is one of the critical numbers.
We will discuss the x-intercepts in Section 4.8.

b) $f(x) = \dfrac{x^2}{3(x+4)^2}$

c) $f(x) = \dfrac{(x-1)^2}{x^2+1}$
In c), give the y-coordinates of any inflection points (IPs) as integers or as decimals rounded off to two decimal places.

d) $f(x) = \sqrt[3]{x^2}\left[3 - 2\left(\sqrt[3]{x}\right)\right]$. Hint: Rewrite by multiplying first!

2) Sketch the graph of $y = \left| \frac{x^2 - 16}{x - 4} \right|$. Indicate all features of interest.

   Hint: Simplify first.

3) Draw the graph of a function f with the following properties:

   $f(0) = 2$,

   $f(3) = 4$,

   $f(5) = 6$,

   $f'(0)$ does not exist (DNE) (or is "undefined"),

   $f'(5) = 0$,

   $f'(x) < 0$ wherever $x < 0$ or $x > 5$,

   $f'(x) > 0$ wherever $0 < x < 5$,

   $f''(0)$ does not exist (DNE) (or is "undefined"),

   $f''(3) = 0$,

   $f''(x) < 0$ wherever $x < 0$ or $x > 3$, and

   $f''(x) > 0$ wherever $0 < x < 3$.

   There are infinitely many different possible graphs that will work here.

SECTION 4.6: OPTIMIZATION

Give exact measurements (with the possible exception of #1), and round off irrational results to four significant digits.

1) (Closed box problem). We need a closed rectangular cardboard box with a square top, a square bottom, and a volume of 32 m^3. Find the dimensions of the valid box that requires the least amount of cardboard, and find the amount of cardboard needed. (Ignore the thickness of the cardboard.) Round off measurements to four significant digits. **ADDITIONAL PROBLEM:** Give the exact measurements.

2) (Open box problem). Repeat Exercise 1, except the box must have an open top. Explain why the optimal dimensions here are different compared to Exercise 1.

3) (Aquarium problem). A glass aquarium is to be shaped as a right circular cylinder with an open top and a capacity of two cubic meters. Find the dimensions of the valid cylinder that requires the least amount of glass, and find that amount of glass. (Ignore the thickness of the glass.) How would the diameter compare to the height?

4) (Six pigpen problem). A 2×3 array of six congruent rectangular pigpens (that all look the same from above) will be in the overall shape of a rectangle R. We may use 100 feet of fencing to form the boundaries of the pigpens. Find the dimensions for a single pigpen that will maximize the total area of all the pigpens, and find this total area. (The fencing separating the pigpens has constant height, so we may ignore height in our calculations. Also, assume the boundaries between pigpens are not double-fenced; that is, assume that the thickness of the fencing between pigpens is the same as the thickness of the fencing along the outer boundary, R.)

5) (Chase problem). At midnight, you are 60 feet due north of Jonas. You run due east at ten feet per second. Jonas walks two feet per second due north. How many seconds after midnight is Jonas closest to you? What is the corresponding minimum distance separating you and Jonas?

6) (Cheap building problem). We need to build a building in the shape of a rectangular box with a capacity of 500,000 cubic feet. We require the length of the floor to be twice the width. The floor will cost \$3 per square foot, the vertical walls will cost \$4 per square foot, and the ceiling will cost \$5 per square foot. To the nearest hundredth of a foot, what are the dimensions of the cheapest building that we can build? What is that building's cost to the nearest dollar?

7) (Closest point problem). A military aircraft flies along the parabola $y = x^2 + 1$ in the usual Cartesian xy-plane. (Distance is measured in meters.) A UFO is hovering at the point $(6, 4)$. Find the point on the parabola that is closest to the UFO, and find the corresponding minimum distance.

8) (Big squares problem). Prove that, among all rectangles with fixed perimeter p, where $p > 0$, the largest in area is a square.

SECTION 4.7: MORE APPLICATIONS OF DERIVATIVES

1) The position function s of a particle moving along a coordinate line is given by $s(t) = 4t^3 + 15t^2 - 18t + 1$, where time t is measured in minutes, and $s(t)$ is measured in feet. (We saw this function in Section 3.2, Exercise 6.) Assume the positive direction extends to the right and the negative direction extends to the left.
   a) Determine the velocity function [rule] $v(t)$.
   b) Determine the time intervals in which the particle moves to the right. Use a sign chart.
   c) Determine the time intervals in which the particle moves to the left. Use a sign chart.
   d) Draw a schematic representing the motion of the particle in the time interval $[-6, 3]$, as we have done in the notes.
   e) Determine the acceleration function [rule] $a(t)$.
   f) Evaluate $v(-4)$ and $a(-4)$. At time $t = -4$ (minutes), is the particle moving to the right or to the left? Is it speeding up or slowing down? (More precisely, we are referring to an open interval containing $t = -4$ minutes. The same idea goes for g), h), and i) below.)
   g) Evaluate $v(-2)$ and $a(-2)$. At time $t = -2$ (minutes), is the particle moving to the right or to the left? Is it speeding up or slowing down?
   h) Evaluate $v(0)$ and $a(0)$. At time $t = 0$ (minutes), is the particle moving to the right or to the left? Is it speeding up or slowing down?
   i) Evaluate $v(1)$ and $a(1)$. At time $t = 1$ (minutes), is the particle moving to the right or to the left? Is it speeding up or slowing down?

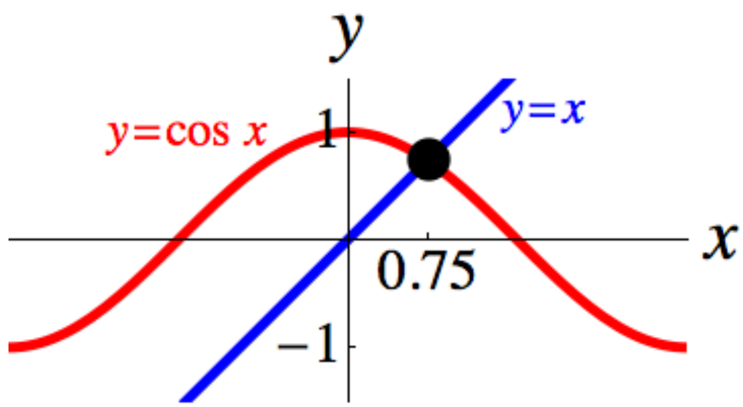
2) Assume that, if we produce x units of a device, we sell all x devices. We sell each device for \$200, so the revenue function is modeled by $R(x) = 200x$. The cost function is modeled by $C(x) = 3x^2 + 500$.
   a) Find the profit function [rule], $P(x)$.
   b) Find the marginal profit at $x = 30$ devices. Based on this result alone, would you be inclined to increase or decrease production?
   c) Find the optimal number of devices to produce.

SECTION 4.8: NEWTON'S METHOD

1) Use Newton's Method to approximate $\sqrt[3]{7}$ to four decimal places.
   Use $x_1 = 2$ as your seed. Round off intermediate iterates to five decimal places.

2) Let $f(x) = -x^4 + 4x^3 + 48x^2 + 112x - 500$.
   We saw this in Section 4.5, Exercise 1a. We believe that the graph of $y = f(x)$ has an x-intercept at about 10. (Look at the graph in the Answers, Section 4.5.)
   Use Newton's Method to approximate this x-intercept to two decimal places.
   Use $x_1 = 10$ as your seed. Round off intermediate iterates to three decimal places.
   Note: There do exist quartic formulas for finding the zeros of f exactly, but they are very involved!

3) Use Newton's Method to approximate a real solution to the equation $\cos x = x$ to four decimal places. Judging from the graphs of $y = \cos x$ and $y = x$ below, use $x_1 = 0.75$ as your seed; it appears that the equation has only one real solution. Round off intermediate iterates to five decimal places.



4) (A failure of Newton's Method.) Let $f(x) = \sqrt[3]{x}$. We know that the only real zero of f is 0. Begin with the seed $x_1 = 1$. Use Newton's Method to obtain x_2 and x_3. (Do you believe that further iterates will approach 0? It will help to remember what the graph of the function looks like.) How are the tangent lines changing?

CHAPTER 4: APPLICATIONS OF DERIVATIVES

SECTION 4.1: EXTREMA

1) a) A.Max Value: 23, A.Max Point: $(2, 23)$; A.Min Value: 5; A.Min Point: $(-1, 5)$.

   b) A.Max Value: 10, A.Max Point: $(0, 10)$;

   A.Min Value: $-\dfrac{34}{3}$; A.Min Point: $\left(2, -\dfrac{34}{3}\right)$.

   c) A.Max Value: 20, A.Max Point: $(-4, 20)$;

   A.Min Value: 12; A.Min Point: $(-2, 12)$.

2) a) $\text{Dom}(f) = (-\infty, \infty)$; CNs: -2, $\dfrac{3}{16}$, and 2. Hint: Try Factoring by Grouping.

   b) $\text{Dom}(g) = (-\infty, -6] \cup [6, \infty)$; CNs: -6 and 6.

   c) $\text{Dom}(h) = \left[\dfrac{1}{4}, \infty\right)$; CN: $\dfrac{1}{4}$.

   d) $\text{Dom}(p) = (-\infty, \infty)$;

   CNs: $\left\{ \theta \in \mathbb{R} \;\middle|\; \theta = \dfrac{\pi}{2} + \pi n, \text{ or } \theta = \dfrac{\pi}{6} + 2\pi n, \text{ or } \theta = \dfrac{5\pi}{6} + 2\pi n \quad (n \in \mathbb{Z}) \right\}$;

   Hint: After differentiating, use a Double-Angle ID.

   e) $\text{Dom}(q) = \left\{ x \in \mathbb{R} \;\middle|\; x \neq \pi n \quad (n \in \mathbb{Z}) \right\}$; CNs: NONE.

SECTION 4.2: MEAN VALUE THEOREM (MVT) FOR DERIVATIVES

1) a) f satisfies the hypotheses on $[1, 5]$; $c = 3$.

   b) f does not satisfy the hypotheses on $[3, 7]$, because $f(3) \neq f(7)$.

   c) f satisfies the hypotheses on $[-6, -1]$; $c = -5$, or $c = -\dfrac{7}{2} = -3.5$, or $c = -2$.

   d) f does not satisfy the hypotheses on $[-4, 4]$, because f is not differentiable at 0, and $0 \in (-4, 4)$; therefore, f is not differentiable on $(-4, 4)$.

2) a) f satisfies the hypotheses on $\left[1,4\right]$; $c=2$.

Note 1: $-2\notin\left[1,4\right]$. Note 2: Rolle's Theorem also applies!

b) f satisfies the hypotheses on $\left[-2,3\right]$; $c=\dfrac{-5+\sqrt{139}}{6}=\dfrac{\sqrt{139}-5}{6}\approx 1.1316$.

Note: $\dfrac{-5-\sqrt{139}}{6}=-\dfrac{5+\sqrt{139}}{6}\approx -2.7983$, so $\dfrac{-5-\sqrt{139}}{6}\notin\left[-2,3\right]$.

c) f does not satisfy the hypotheses on $\left[-8,8\right]$, because f is not differentiable at 0, and $0\in\left(-8,8\right)$; therefore, f is not differentiable on $\left(-8,8\right)$.

d) f satisfies the hypotheses on $\left[0,2\right]$; all real values in $\left(0,2\right)$ satisfy the theorem. (Can you see graphically why this is true?)

SECTION 4.3: FIRST DERIVATIVE TEST

1) a) $\mathrm{Dom}\left(f\right)=\left(-\infty,\infty\right)$. f is odd, so its graph is symmetric about the origin.

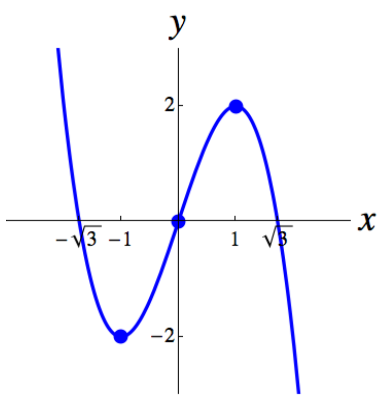
y-intercept: 0, or $\left(0,0\right)$. x-intercepts: $\left(0,0\right)$, $\left(\sqrt{3},0\right)$, $\left(-\sqrt{3},0\right)$.

Holes: None. VAs: None. HAs: None. SAs: None.

Points at critical numbers:

$\left(-1,-2\right)$, a local minimum point; $\left(1,2\right)$, a local maximum point;

f is increasing on $\left[-1,1\right]$. f is decreasing on $\left(-\infty,-1\right],\left[1,\infty\right)$.



b) $\mathrm{Dom}\left(f\right)=\left(-\infty,\infty\right)$. f is neither even nor odd.

y-intercept: -5, or $\left(0,-5\right)$. Holes: None. VAs: None. HAs: None. SAs: None.

Points at critical numbers: $\left(-5,-105\right)$, a local minimum point;

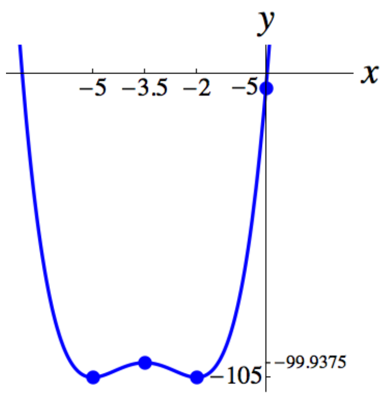
$\left(-\dfrac{7}{2},-\dfrac{1599}{16}\right)$, or $\left(-3.5,-99.9375\right)$, a local maximum point;

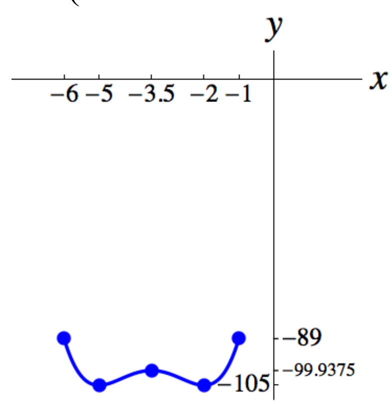
$\left(-2,-105\right)$, a local minimum point.

f is increasing on $\left[-5, -\frac{7}{2}\right], \left[-2, \infty\right)$; or $\left[-5, -3.5\right], \left[-2, \infty\right)$.

f is decreasing on $\left(-\infty, -5\right], \left[-\frac{7}{2}, -2\right]$, or $\left(-\infty, -5\right], \left[-3.5, -2\right]$.

Looking back at Section 4.2: (Axes are scaled differently.)





c) $\text{Dom}(f) = (-\infty, 4) \cup (4, \infty)$. f is neither even nor odd.

y-intercept: $-\frac{1}{4}$, or $\left(0, -\frac{1}{4}\right)$. Holes: None.

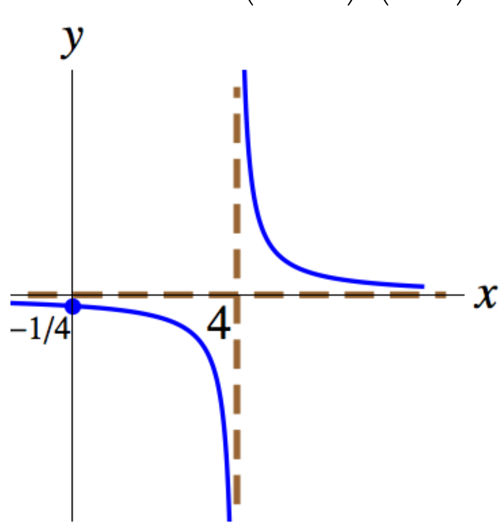
VA: $x = 4$, because $\lim_{x \to 4^+} f(x) = \infty$ (or because $\lim_{x \to 4^-} f(x) = -\infty$).

HA: only $y = 0$, because $\lim_{x \to \infty} f(x) = 0$, and $\lim_{x \to -\infty} f(x) = 0$.

SAs: None.

Points at critical numbers: None.

f is decreasing on $(-\infty, 4), (4, \infty)$.



d) $\text{Dom}(f) = (-2\pi, 2\pi)$. f is odd, so its graph is symmetric about the origin.

y-intercept: 0, or $(0,0)$.

Holes: None, not counting the excluded endpoints of the graph.
VAs: None. HAs: None. SAs: None.
Points at critical numbers:

$A\left(-\dfrac{5\pi}{3}, \dfrac{5\pi + 3\sqrt{3}}{6}\right)$, a local maximum point;

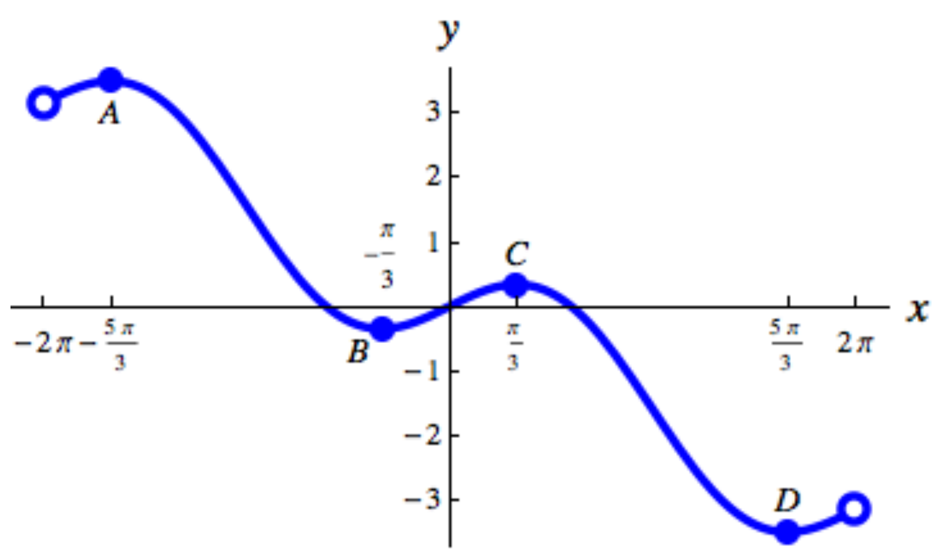
$B\left(-\dfrac{\pi}{3}, \dfrac{\pi - 3\sqrt{3}}{6}\right)$, a local minimum point;

$C\left(\dfrac{\pi}{3}, \dfrac{3\sqrt{3} - \pi}{6}\right)$, a local maximum point (can use B; f is odd);

$D\left(\dfrac{5\pi}{3}, -\dfrac{5\pi + 3\sqrt{3}}{6}\right)$, a local minimum point (can use A; f is odd).

f is increasing on $\left(-2\pi, -\dfrac{5\pi}{3}\right], \left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right], \left[\dfrac{5\pi}{3}, 2\pi\right)$.

f is decreasing on $\left[-\dfrac{5\pi}{3}, -\dfrac{\pi}{3}\right], \left[\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right]$.



2) a) A local maximum point;

b) Yes, if f lost differentiability at a number besides 7, then, because it is in $\text{Dom}(f)$, that other number would be a critical number. Contradiction!

SECTION 4.4: SECOND DERIVATIVES

1) a) PIN: 0. Concave up on $\left(-\infty, 0\right]$. Concave down on $\left[0, \infty\right)$.

PIN corresponds to IP: $\left(0, 0\right)$.

b) PINs: Both of $\dfrac{-7 \pm \sqrt{3}}{2}$; these are about -2.634 and -4.366.

Concave up on $\left(-\infty, \dfrac{-7-\sqrt{3}}{2}\right], \left[\dfrac{-7+\sqrt{3}}{2}, \infty\right)$, about

$\left(-\infty, -4.366\right], \left[-2.364, \infty\right)$.

Concave down on $\left[\dfrac{-7-\sqrt{3}}{2}, \dfrac{-7+\sqrt{3}}{2}\right]$, about $\left[-4.366, -2.364\right]$.

Both PINs correspond to IPs.

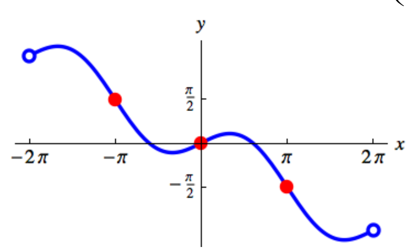
c) PINs: None; observe that 4 is not in $\mathrm{Dom}\left(f\right)$.

Concave up on $\left(4, \infty\right)$. Concave down on $\left(-\infty, 4\right)$. IPs: None.

d) PINs: $-\pi$, 0, and π.

Concave up on $\left[-\pi, 0\right], \left[\pi, 2\pi\right)$. Concave down on $\left(-2\pi, -\pi\right], \left[0, \pi\right]$.

All PINs correspond to IPs: $\left(-\pi, \dfrac{\pi}{2}\right)$, $\left(0, 0\right)$, and $\left(\pi, -\dfrac{\pi}{2}\right)$; see red points.



2) Hints: Verify that $f'\left(-5\right) = 0$, and show that $f''\left(-5\right) > 0$.

3) Hints: A Power-Reducing trig ID will prove very helpful here.
$g\left(\theta\right) = 2 + 2\cos\left(6\theta\right)$. $g'\left(\theta\right) = -12\sin\left(6\theta\right)$. $g''\left(\theta\right) = -72\cos\left(6\theta\right)$.

a) It is a local maximum point, because $g'\left(\dfrac{\pi}{3}\right) = 0$, and $g''\left(\dfrac{\pi}{3}\right) < 0$.

b) Nothing, because $g'\left(\dfrac{\pi}{4}\right) \neq 0$.

4) Nothing, because $h''\left(0\right) = 0$.

5) Employment was decreasing but at a slower and slower rate.

SECTION 4.5: GRAPHING

1) a) $\text{Dom}(f) = (-\infty, \infty)$.

f is neither even nor odd.

y-intercept: -500, or $(0, -500)$. x-intercepts: We will discuss in Section 4.8.

Holes: None. VAs: None. HAs: None. SAs: None.

$f'(x) = -4x^3 + 12x^2 + 96x + 112$.

CNs: -2 and 7. Points at critical numbers:

$(-2, -580)$, neither a local maximum nor a local minimum point;

$(7, 1607)$, a local maximum point.

f is increasing on $(-\infty, 7]$.

f is decreasing on $[7, \infty)$.

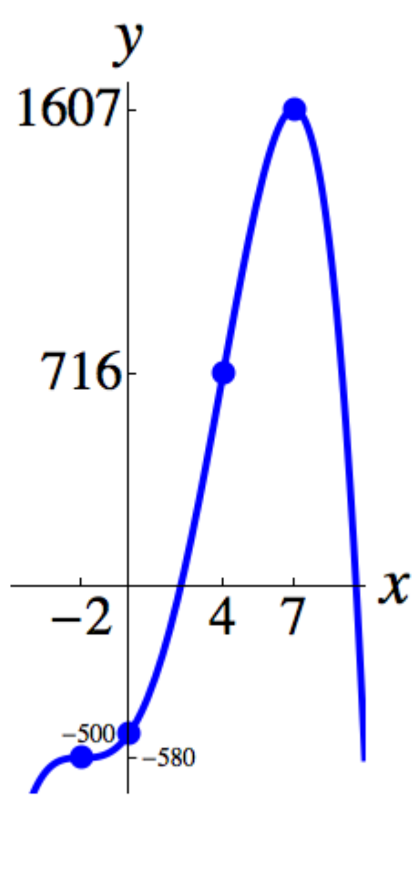
$f''(x) = -12x^2 + 24x + 96$.

PINs: -2 and 4.

Concave up on $[-2, 4]$.

Concave down on $(-\infty, -2], [4, \infty)$.

Both PINs correspond to IPs: $(-2, -580)$ and $(4, 716)$.



(Axes are scaled differently.)

b) $\mathrm{Dom}(f) = (-\infty, -4) \cup (-4, \infty)$.

f is neither even nor odd.

y-intercept: 0, or $(0, 0)$. x-intercept: 0, or $(0, 0)$.

Holes: None.

VA: $x = -4$, because $\lim\limits_{x \to -4^+} f(x) = \infty$ (or because $\lim\limits_{x \to -4^-} f(x) = \infty$).

HA: only $y = \dfrac{1}{3}$, because $\lim\limits_{x \to \infty} f(x) = \dfrac{1}{3}$, and $\lim\limits_{x \to -\infty} f(x) = \dfrac{1}{3}$.

SAs: None.

$$f'(x) = \frac{8x}{3(x+4)^3}.$$

CN: 0. Points at critical numbers:

$(0, 0)$, a local minimum point.

f is increasing on $(-\infty, -4), [0, \infty)$.

f is decreasing on $(-4, 0]$.

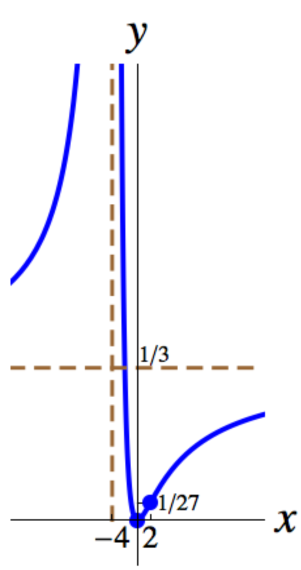
$$f''(x) = \frac{16(2-x)}{3(x+4)^4}.$$

PIN: 2.

Concave up on $(-\infty, -4), (-4, 2]$.

Concave down on $[2, \infty)$.

The PIN does correspond to an IP: $\left(2, \dfrac{1}{27}\right)$.



(Axes are scaled differently.)

c) $\text{Dom}(f) = (-\infty, \infty)$.

f is neither even nor odd.

y-intercept: 1, or $(0, 1)$. x-intercept: 1, or $(1, 0)$.

Holes: None. VAs: None.

HA: only $y = 1$, because $\lim_{x\to\infty} f(x) = 1$, and $\lim_{x\to-\infty} f(x) = 1$.

SAs: None.

$$f'(x) = \frac{2\left(x^2 - 1\right)}{\left(x^2 + 1\right)^2}.$$

CNs: -1 and 1. Points at critical numbers:

$(-1, 2)$, a local maximum point.

$(1, 0)$, a local minimum point.

f is increasing on $(-\infty, -1], [1, \infty)$.

f is decreasing on $[-1, 1]$.

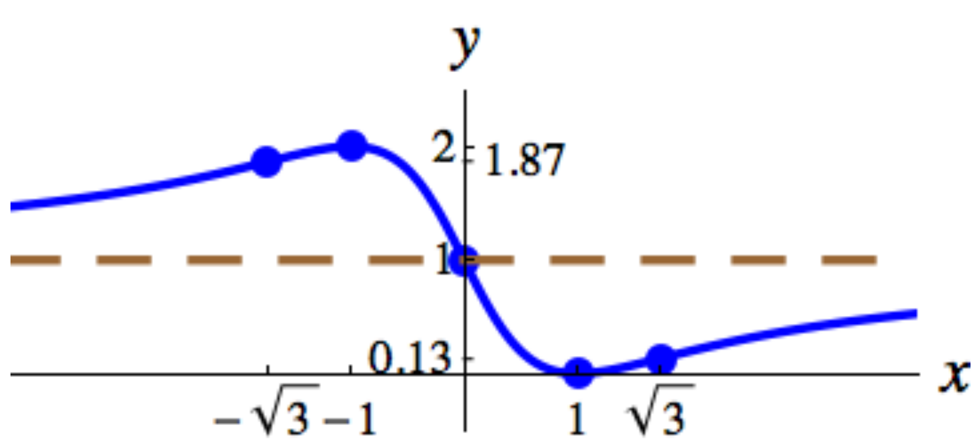
$$f''(x) = \frac{4x\left(3 - x^2\right)}{\left(x^2 + 1\right)^3}.$$

PINs: $-\sqrt{3}$, 0, and $\sqrt{3}$.

Concave up on $\left(-\infty, -\sqrt{3}\right], \left[0, \sqrt{3}\right]$.

Concave down on $\left[-\sqrt{3}, 0\right], \left[\sqrt{3}, \infty\right)$.

The PINs correspond to IPs: $\left(-\sqrt{3}, \text{about } 1.87\right)$, $(0, 1)$, and $\left(\sqrt{3}, \text{about } 0.13\right)$.



d) $\text{Dom}(f) = (-\infty, \infty)$.

f is neither even nor odd.

y-intercept: 0, or $(0, 0)$. x-intercepts: 0 and $\frac{27}{8} = 3.375$, or $(0, 0)$ and $(3.375, 0)$.

Holes: None. VAs: None. HAs: None. SAs: None.

$f'(x) = 2x^{-1/3} - 2$, or $\dfrac{2\left(1 - \sqrt[3]{x}\right)}{\sqrt[3]{x}}$.

CNs: 0 and 1. Points at critical numbers:

$(0, 0)$, a local minimum point.

$(1, 1)$, a local maximum point.

f is increasing on $[0, 1]$.

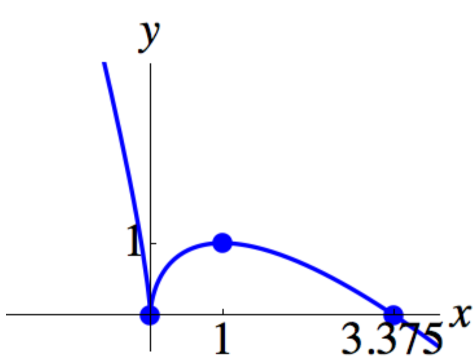
f is decreasing on $(-\infty, 0], [1, \infty)$.

$f''(x) = -\dfrac{2}{3x^{4/3}}$.

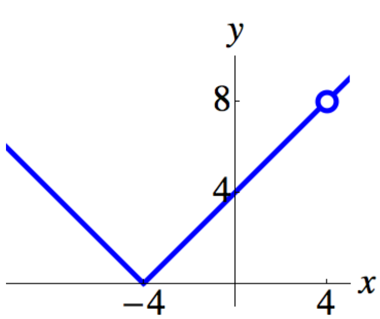
PIN: 0.

Concave down on $(-\infty, 0], [0, \infty)$.

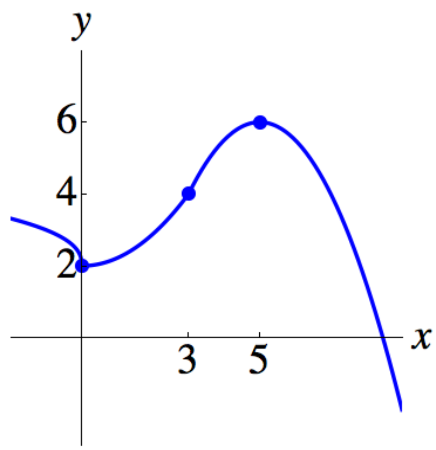
The PIN does not correspond to an IP: no IPs.



2) Make sure to indicate the hole at $(4, 8)$ and the local minimum (and corner) point at $(-4, 0)$.



3) The following is one of infinitely many possible graphs:



SECTION 4.6: OPTIMIZATION

1) We want a cube of side length $2\left(\sqrt[3]{4}\right)\text{ m} = 2\left(2^{2/3}\right)\text{ m} = 2^{5/3}\text{ m} \approx 3.175\text{ m}$.

   It requires $48\left(\sqrt[3]{2}\right)\text{ m}^2 = 48\left(2^{1/3}\right)\text{ m}^2 \approx 60.48\text{ m}^2$ of cardboard.

   Hints: If x is the side length of the square top (or bottom) and y is the height of the box, then surface area $S = 2x^2 + 4xy = 2x^2 + \dfrac{128}{x}$, which is continuous on $(0, \infty)$.

   $S' < 0$ on $\left(0,\ 2\left(\sqrt[3]{4}\right)\right)$, and $S' > 0$ on $\left(2\left(\sqrt[3]{4}\right),\ \infty\right)$; this verifies that S has an absolute minimum at $x = 2\left(\sqrt[3]{4}\right)\text{ m}$.

2) Optimal dimensions: 4 m $\times$ 4 m $\times$ 2 m. The box requires 48 m^2 of cardboard. The absence of a top side favors a larger bottom side and allows for a smaller total surface area. (Compare to the pigpen problems in the notes.)

   Hint: Using the notation from Exercise 1, $S = x^2 + 4xy = x^2 + \dfrac{128}{x}$.

3) Base radius $r = \sqrt[3]{\dfrac{2}{\pi}}\text{ m} \approx 0.8603\text{ m}$, and height $h = \sqrt[3]{\dfrac{2}{\pi}}\text{ m} \approx 0.8603\text{ m}$.

   Hint: Surface area $S = \pi r^2 + 2\pi rh = \pi r^2 + \dfrac{4}{r}$.

   The aquarium requires $3\left(\sqrt[3]{4\pi}\right)\text{ m}^2 = 3\left(2^{2/3}\right)\left(\pi^{1/3}\right)\text{ m}^2 \approx 6.975\text{ m}^2$ of glass.

   (It's easier to use $S = \pi r^2 + 2\pi rh$ instead of $S = \pi r^2 + \dfrac{4}{r}$ to find this.)

   The diameter would be twice the height, so the aquarium would be "squat."

4) $x = \frac{50}{9}$ ft (by) $y = \frac{25}{4}$ ft, or $5\frac{5}{9}$ ft by $6\frac{1}{4}$ ft, where R has dimensions $3x$ by $2y$.

The total area (enclosed by R) is $\frac{625}{3}\text{ ft}^2 = 208\frac{1}{3}\text{ ft}^2$.

Hint: If R has dimensions $3x$ by $2y$, then total area $A = 6xy = 75x - \frac{27}{4}x^2$.

5) $\frac{15}{13}\text{ sec} = 1\frac{2}{13}\text{ sec} \approx 1.154\text{ sec}$. The corresponding minimum distance is $\sqrt{\frac{45{,}000}{13}}\text{ ft} = \frac{150\sqrt{26}}{13}\text{ ft} \approx 58.83\text{ ft}$, which is just a bit less than the initial 60 ft.

6) About 57.24 feet (floor width w) by 114.47 feet (length l) by 76.31 feet (height).
The corresponding cost is about \$157,244.

Hint: Cost $C = 3lw + 4(2lh) + 4(2wh) + 5lw = 16w^2 + \frac{6{,}000{,}000}{w}$.

7) Point: $(2, 5)$. The corresponding minimum distance is $\sqrt{17}\text{ m} \approx 4.123\text{ m}$.

Hint 1: Minimize the [square of the] distance between points of the form $(x, x^2 + 1)$ and the point $(6, 4)$.

Hint 2: Remember the Rational Zero Test and Synthetic Division.
See Sections 2.3 and 2.5 in the Precalculus notes.

Note: We also get integers for the coordinates of the closest point on the parabola if the UFO is at $(3,1)$, $(-3,1)$, or $(10,3)$, among others.

8) Hint: Set up a generic rectangle with dimensions l and w. Show that $l = w$ for the largest rectangle.

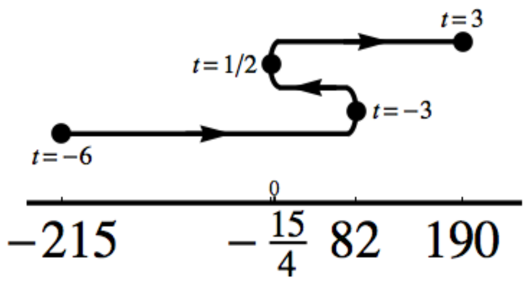
SECTION 4.7: MORE APPLICATIONS OF DERIVATIVES

1) a) $v(t) = 12t^2 + 30t - 18$

b) $(-\infty, -3), \left(\frac{1}{2}, \infty\right)$

c) $\left(-3, \frac{1}{2}\right)$

d)



e) $a(t) = 24t + 30$

f) $v(-4) = 54$, $a(-4) = -66$, moving to the right, slowing down

g) $v(-2) = -30$, $a(-2) = -18$, moving to the left, speeding up

h) $v(0) = -18$, $a(0) = 30$, moving to the left, slowing down

i) $v(1) = 24$, $a(1) = 54$, moving to the right, speeding up

2) a) $P(x) = -3x^2 + 200x - 500$

b) $P'(30) = 20 \dfrac{\$}{\text{device unit}}$, increase production.

c) 33 devices. The CN is $\dfrac{100}{3} = 33\dfrac{1}{3}$ devices, and the absolute maximum of P is there if the domain is taken to be $[0, \infty)$. However, an integer number of devices such as 33 or 34 devices would be a more appropriate answer to this problem. $P(33) = \$2833$, and $P(34) = \$2832$, so $P(33) > P(34)$, and 33 devices is a better production level than 34 devices.

SECTION 4.8: NEWTON'S METHOD

1) $x_2 \approx 1.91667$, $x_3 \approx 1.91294$, $x_4 \approx 1.91293$. $\sqrt[3]{7} \approx 1.9129$.

2) $x_2 \approx 9.664$, $x_3 \approx 9.632$, $x_4 \approx 9.631$. Answer: about 9.63.

3) $x_2 \approx 0.73911$, $x_3 \approx 0.73909$. Answer: about 0.7391. Hint 1: Make sure your calculator is in radian mode! Hint 2: Isolate 0 on one side of the given equation.

4) $x_2 = -2$, $x_3 = 4$. (Note: In fact, the iterates will move further away from 0.) The tangent lines are getting flatter and flatter; that is, the derivatives at our iterates are getting closer to 0. (Note: In the computational field of numerical analysis, derivatives that are close to zero can lead to unstable results.)

CHAPTER 5: INTEGRALS

SECTION 5.1: ANTIDERIVATIVES and INDEFINITE INTEGRALS

1) Evaluate the following indefinite integrals. Assume that all integrands are defined [and continuous] "where we care."

a) $\int \left(2x^3 - \frac{\sqrt[4]{x^3}}{5} - \frac{x^5}{2} + \frac{2}{3\sqrt{x}} - 3 \right) dx$

b) $\int y\sqrt{y}\, dy$

c) $\int \frac{3t^3 - 2t^2 + \sqrt{t}}{t}\, dt$

d) $\int (w+3)(w+4)\, dw$

e) $\int (3z)^4\, dz$

f) $\int \frac{\left(t^2+3\right)^2}{t^6}\, dt$. Your final answer must not contain negative exponents.

g) $\int \frac{x^3+8}{x+2}\, dx$

h) $\int (3\sin x + 5\cos x)\, dx$

i) $\int \frac{4}{\cos^2\theta}\, d\theta$

j) $\int (\sec t)(\csc t)(\cot t)\, dt$

k) $\int \frac{\sin r}{\cos^2 r}\, dr$

l) $\int (\cot\theta)(1+\cot^2\theta)(\sin\theta)\, d\theta$

m) $\int \pi^2\, dx$

n) $\int \sin^5 \pi\, dx$

o) $\int \left(a^{10} + abt\right) dt$, if a and b represent real constants

p) $\int D_x\left[\tan^5\left(x^4\right)\right] dx$

2) Evaluate $D_x\left(\int \sqrt{x^5 + x}\, dx\right)$.

3) For each part below, solve the differential equation subject to the given conditions.

   a) $f'(x) = 6x^2 + 2x - 1$ subject to: $f(-2) = 30$

   b) $\dfrac{dy}{dx} = 2\sqrt{x}$ subject to: $y = 34$ if $x = 9$

   c) $f''(x) = 3x + 2$ subject to: $f'(1) = \dfrac{1}{2}$ and $f(-1) = 7$

   d) $\dfrac{d^2 y}{dx^2} = 7\sin x + 2\cos x$ subject to: both ($y = 10$ and $y' = 4$) if $x = 0$

4) Assuming that a particle is moving on a coordinate line with acceleration $a(t) = 2 - 6t$ and with initial conditions $v(0) = -5$ and $s(0) = 4$, find the position function rule $s(t)$ for the particle. $s(t)$ is measured in feet, t in minutes, $v(t)$ in feet per minute, and $a(t)$ in feet per minute per minute.

5) On Earth, a projectile is fired vertically upward from ground level with a velocity of 1600 feet per second. Ignore air resistance. Use -32 feet per second per second as the Earth's [signed] gravitational constant of acceleration.

   a) Find the projectile's height $s(t)$ above ground t seconds after it is fired. (The formula will be relevant until the time the projectile hits the ground.)

   b) Find the projectile's maximum height.

6) A brick is thrown directly downward from a height of 96 feet with an initial velocity of 16 feet per second. Use -32 feet per second per second as the Earth's [signed] gravitational constant of acceleration.

   a) Find the brick's height above ground t seconds after it is thrown. (Your formula will be relevant up until the time at which the brick hits the ground.)

   b) Find how long it takes for the brick to hit the ground (starting from the time the brick is thrown).

   c) Find the velocity at which the brick hits the ground.

7) A country has natural gas reserves of 250 billion cubic feet. If the gas is consumed at the rate of $3 + 0.02t$ billion cubic feet per year (where $t = 0$ corresponds to now, and t is measured in years), how long will it be before the reserves are depleted? Round off your answer to four significant digits.

SECTION 5.2: *u* SUBSTITUTIONS

1) Evaluate the following indefinite integrals. In this section, assume that all integrands are defined [and continuous] "where we care."

   a) $\int 3x\left(x^2+5\right)^3\,dx$

   b) $\int \sqrt{4p-5}\,dp$

   c) $\int \alpha^4 \sin\left(\pi - 2\alpha^5\right)\,d\alpha$

   d) $\int \dfrac{2r^3+r}{\sqrt{r^4+r^2}}\,dr$

   e) $\int \left(\sin x\right)\left(\sec^5 x\right)\,dx$

   f) $\int \left[\sin\left(3\theta\right)+\theta\right]\,d\theta$

   g) $\int \dfrac{\sec^2\left(\sqrt{x}\right)}{\left(\sqrt{x}\right)\tan^3\left(\sqrt{x}\right)}\,dx$

   h) $\int \dfrac{1}{x^2+2x+1}\,dx$

   i) $\int \dfrac{\sec\beta\tan\beta}{4+12\sec\beta+9\sec^2\beta}\,d\beta$

   j) $\int \dfrac{x}{\sqrt{x+1}}\,dx$

2) Use "Guess-and-Check" to evaluate the following indefinite integrals.

   a) $\int \cos\left(4x\right)\,dx$; b) $\int \sin\left(3\theta\right)\,d\theta$; c) $\int \sec^2\left(\dfrac{\theta}{2}\right)\,d\theta$; d) $\int \sec\left(2\alpha\right)\tan\left(2\alpha\right)\,d\alpha$;

   e) $\int \csc\left(7\alpha\right)\cot\left(7\alpha\right)\,d\alpha$

3) We will evaluate $\int \dfrac{\left(\sqrt{x}-1\right)^2}{\sqrt{x}}\,dx$ in two different ways.

   a) Evaluate $\int \dfrac{\left(\sqrt{x}-1\right)^2}{\sqrt{x}}\,dx$ **without** using a substitution.

   b) Evaluate $\int \dfrac{\left(\sqrt{x}-1\right)^2}{\sqrt{x}}\,dx$ by using a substitution.

   c) Show that your answers to a) and b) are equivalent. (This is primarily an exercise in algebra.)

ADDITIONAL PROBLEMS (#4-6)

4) Evaluate $\int \left(2+\dfrac{1}{\sqrt{z}}\right)^4 \cdot \dfrac{1}{z\sqrt{z}}\,dz$.

5) Evaluate $\int \left(u^2+1\right)\sqrt{u-2}\,du$. Note: Do not use u as your substitution variable.

6) Evaluate $\int \dfrac{x^2+2x}{x^2+2x+1}\,dx$.

SECTIONS 5.3/5.4: AREA and DEFINITE INTEGRALS

1) Assume that $\int_1^4 \sqrt{x}\, dx = \frac{14}{3}$. Evaluate the following integrals based on this information.
   a) $\int_1^4 \sqrt{t}\, dt$
   b) $\int_4^4 \sqrt{x}\, dx + \int_4^1 \sqrt{x}\, dx$

2) Evaluate the following definite integrals using areas.
   a) $\int_{-1}^5 6\, dx$
   b) $\int_{-3}^2 (2x+6)\, dx$
   c) $\int_0^3 |x-1|\, dx$. We will revisit this problem in Section 5.6.
   d) $\int_0^3 \sqrt{9-x^2}\, dx$
   e) $\int_{-2}^2 \left(3+\sqrt{4-x^2}\right) dx$

3) We will approximate $\int_1^5 (2x+3)\, dx$ using Riemann approximations and the partition P of the interval $[1, 5]$, where P is given by: $\{1, 3, 4, 5\}$.
   a) Use a Left-Hand Riemann Approximation.
   b) Use a Right-Hand Riemann Approximation.
   c) Use a Midpoint Riemann Approximation.
   d) Use an area argument to find the exact value of the integral.

4) Approximate $\int_0^6 \left(8-\frac{1}{2}x^2\right) dx$ by using a Midpoint Riemann Approximation and a regular partition of the interval $[0, 6]$ into six subintervals.

SECTION 5.5: PROPERTIES OF DEFINITE INTEGRALS

1) Assuming that $\int_1^4 x^2\,dx = 21$ and $\int_1^4 x\,dx = \frac{15}{2}$, evaluate the following definite integrals.
   a) $\int_1^4 \left(3x^2 + 5\right) dx$
   b) $\int_4^1 \left(2 - 9t - 4t^2\right) dt$

2) True or False: $\int_0^{4\pi} \left(1 + \sin x\right) dx \geq 0$. Explain.

3) True or False: $\int_{3\pi}^{5\pi} \left(-\cos^2 x\right) dx \leq 0$. Explain.

4) Express as one integral: $\int_5^1 f(x)\,dx + \int_{-3}^5 f(x)\,dx$.
   Assume that f is everywhere continuous on $\mathbb{R}$.

5) Express as one integral: $\int_c^{c+h} g(t)\,dt - \int_c^h g(t)\,dt$.
   Assume that g is everywhere continuous on $\mathbb{R}$. c and h represent real constants.

6) Let $f(x) = x^2 + 1$. Assume that $\int_{-2}^1 f(x)\,dx = 6$; you will be able to verify this after you study Section 5.6.
   a) Find the average value of f on $[-2, 1]$.
   b) Find a number z that satisfies the conclusion of the Mean Value Theorem (MVT) for Integrals, where the interval of interest is $[-2, 1]$.

7) Let $g(x) = 3\sqrt{x+1}$. Assume that $\int_{-1}^8 g(x)\,dx = 54$; you will be able to verify this after you study Section 5.6.
   a) Find the average value of g on $[-1, 8]$.
   b) Find a number z that satisfies the conclusion of the Mean Value Theorem (MVT) for Integrals, where the interval of interest is $[-1, 8]$. (Consider g instead of f.)

SECTION 5.6: FUNDAMENTAL THEOREM OF CALCULUS (FTC)

1) Evaluate the following definite integrals.

   a) $\int_{-2}^{1} \left(x^2+1\right) dx$. We saw this in Section 5.5, Exercise 6. Yes, the answer is 6.

   b) $\int_{4}^{9} dt$

   c) $\int_{9}^{16} \dfrac{r+2}{\sqrt{r}}\, dr$

   d) $\int_{-8}^{8} \left(\sqrt[3]{x^2}-2\right) dx$. Hint: Do you see how you can reduce your work?

   e) $\int_{4}^{2} \dfrac{t^2-1}{t-1}\, dt$

   f) $\int_{5}^{5} \sin^4\left(\theta^2\right) d\theta$.
   Hint: If you're spending too much time on this, you're doing it wrong!

   g) $\int_{0}^{\pi} 5\sin x\, dx$

   h) $\int_{0}^{\pi/4} \left(\sec t\right)\left(\sec t+\tan t\right) dt$

   i) $\int_{0}^{2} f(x)\, dx$, where $f(x)=\begin{cases} x, & \text{if } 0\le x<1 \\ x^2, & \text{if } 1\le x\le 2 \end{cases}$

2) Evaluate the following definite integrals by using u substitutions (or maybe "Guess-and-Check" if the u sub is linear).

   a) $\int_{-2}^{3} x^2\left(x^3+1\right)^3 dx$

   b) $\int_{1}^{4} \sqrt{5-x}\, dx$

   c) $\int_{0}^{1} \dfrac{1}{(3-2x)^2}\, dx$

   d) $\int_{1}^{4} \dfrac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^3}\, dx$

   e) $\int_{\pi/2}^{\pi} \cos\left(\dfrac{\theta}{3}\right) d\theta$.

   In e), give an exact answer and also give an approximate answer rounded off to three significant digits.

f) $\int_{\pi/4}^{\pi/3} \left[4\sin(2x) + 6\cos(3x) \right] dx$

In f), give an exact answer and also give an approximate answer rounded off to two significant digits.

g) $\int_{0}^{\pi/3} \frac{\sin x}{\cos^2 x}\, dx$

3) We will evaluate $\int_{-2}^{-1} (8-5x)^2\, dx$ in two different ways.

a) First expand $(8-5x)^2$, and then integrate as usual.

b) Use a u substitution.

4) In Sections 5.3/5.4, Exercise 2c, you evaluated $\int_{0}^{3} |x-1|\, dx$ using areas. Now, use the definition of absolute value and evaluate this integral using the FTC.

5) Evaluate $\int_{-\pi}^{\pi} (x+\sin x)\, dx$ by finding an antiderivative and applying the FTC as usual. Why does your answer make sense?

6) Let $f(x) = \frac{x}{\sqrt{x^2+9}}$.

a) Find the average value of f on $[0, 4]$.

b) Find a number z that satisfies the conclusion of the Mean Value Theorem (MVT) for Integrals, where the interval of interest is $[0, 4]$.

7) Find $D_x\left(\int_{\pi/4}^{\pi} \sin^{10} x\, dx \right)$ without performing the integration.

8) Find $D_x\left(\int_{\pi/4}^{x} \sin^{10} t\, dt \right)$ without performing the integration.

SECTION 5.7: NUMERICAL APPROXIMATION OF DEFINITE INTEGRALS

1) We will find approximations for $\int_1^3 \frac{1}{1+x^2}\,dx$ by using a regular partition of the interval $\left[1, 3\right]$ into $n = 4$ subintervals. Round off intermediate results to five decimal places, and round off final answers to four decimal places.
   a) Use the Trapezoidal Rule.
   b) Use Simpson's Rule.

   Note: The exact value of the integral is $\tan^{-1}(3) - \frac{\pi}{4} \approx 0.4636$. You will learn how to work out this integral exactly in Chapter 8.

2) We will find approximations for $\int_0^\pi \sqrt{\sin x}\,dx$ by using a regular partition of the interval $\left[0, \pi\right]$ into $n = 6$ subintervals. Round off intermediate results to five decimal places, and round off final answers to four decimal places.
   a) Use the Trapezoidal Rule.
   b) Use Simpson's Rule.

Note for Kuniyuki's class: If related questions are placed on exams, then you will be given a formula for the Trapezoidal Rule and/or Simpson's Rule, as appropriate.

CHAPTER 5: INTEGRALS

SECTION 5.1: ANTIDERIVATIVES and INDEFINITE INTEGRALS

1)

a) $\frac{x^4}{2} - \frac{4x^{7/4}}{35} - \frac{x^6}{12} + \frac{4\sqrt{x}}{3} - 3x + C$, or $\frac{1}{2}x^4 - \frac{4}{35}x\left(\sqrt[4]{x^3}\right) - \frac{1}{12}x^6 + \frac{4}{3}\sqrt{x} - 3x + C$.
(Note: Many books don't even mention that we require $x > 0$ in the given exercise, even though that restriction is not fully evident in the answer.)

b) $\frac{2y^{5/2}}{5} + C$, or $\frac{2}{5}y^2\sqrt{y} + C$ (The restriction $y \geq 0$ is evident in the answer.)

c) $t^3 - t^2 + 2\sqrt{t} + C$. (Note: We technically require: $t > 0$.)

d) $\frac{w^3}{3} + \frac{7w^2}{2} + 12w + C$, or $\frac{1}{3}w^3 + \frac{7}{2}w^2 + 12w + C$, or $\frac{2w^3 + 21w^2 + 72w}{6} + C$

e) $\frac{81z^5}{5} + C$, or $\frac{81}{5}z^5 + C$

f) $-\frac{1}{t} - \frac{2}{t^3} - \frac{9}{5t^5} + C$, or $C - \frac{1}{t} - \frac{2}{t^3} - \frac{9}{5t^5}$, or $C - \frac{5t^4 + 10t^2 + 9}{5t^5}$

g) $\frac{x^3}{3} - x^2 + 4x + C$. (Note: We technically require: $x \neq -2$.)
Hint: Factor the numerator.

h) $-3\cos x + 5\sin x + C$, or $C - 3\cos x + 5\sin x$

i) $4\tan\theta + C$

j) $-\cot t + C$, or $C - \cot t$. (Note: We technically require: $\cos t \neq 0$.
The restriction $\sin t \neq 0$ is essentially evident in the answer.)

k) $\sec r + C$.

l) $-\csc\theta + C$, or $C - \csc\theta$. Hint: Use a Pythagorean Identity.

m) $\pi^2 x + C$

n) C

o) $a^{10}t + \frac{abt^2}{2} + C$, or $\frac{2a^{10}t + abt^2}{2} + C$, or $\frac{at\left(2a^9 + bt\right)}{2} + C$

p) $\tan^5\left(x^4\right) + C$

2) $\sqrt{x^5 + x}$

3) a) $f\left(x\right) = 2x^3 + x^2 - x + 40$; b) $y = \frac{4}{3}x^{3/2} - 2$, or $y = \frac{4x\sqrt{x} - 6}{3}$

c) $f(x)=\frac{1}{2}x^3+x^2-3x+\frac{7}{2}$, or $f(x)=\frac{x^3+2x^2-6x+7}{2}$

d) $y=-7\sin x-2\cos x+11x+12$

4) $s(t)=t^2-t^3-5t+4$

5) a) $s(t)=-16t^2+1600t$; b) $s(50)=40{,}000$ feet

6) a) $s(t)=-16t^2-16t+96$; b) 2 seconds; c) -80 feet per second

7) The reserves will be depleted in about 67.94 years.

SECTION 5.2: *u* SUBSTITUTIONS

1)

a) $\frac{3(x^2+5)^4}{8}+C$, or $\frac{3}{8}(x^2+5)^4+C$

b) $\frac{(4p-5)^{3/2}}{6}+C$, or $\frac{1}{6}(4p-5)\sqrt{4p-5}+C$

c) $\frac{\cos(\pi-2\alpha^5)}{10}+C$, or $\frac{1}{10}\cos(\pi-2\alpha^5)+C$, or $C-\frac{\cos(2\alpha^5)}{10}$ (the Unit Circle and the Difference Identities give us: $\cos(\pi-\theta)=-\cos\theta$)

d) $\sqrt{r^4+r^2}+C$, or $|r|\sqrt{r^2+1}+C$

e) $\frac{1}{4}\sec^4 x+C$

f) $C-\frac{1}{3}\cos(3\theta)+\frac{1}{2}\theta^2$, or $\frac{3\theta^2-2\cos(3\theta)}{6}+C$

g) $C-\cot^2\left(\sqrt{x}\right)$, or $C-\csc^2\left(\sqrt{x}\right)$ (by Pythagorean IDs)

h) $C-\frac{1}{x+1}$

i) $C-\frac{1}{3(2+3\sec\beta)}$

j) $\frac{2}{3}(x+1)^{3/2}-2\sqrt{x+1}+C$, or $\frac{2\sqrt{x+1}\,(x-2)}{3}$. Hint: $u=x+1 \Rightarrow x=u-1$.

2) a) $\frac{1}{4}\sin(4x)+C$; b) $C-\frac{1}{3}\cos(3\theta)$; c) $2\tan\left(\frac{\theta}{2}\right)+C$; d) $\frac{1}{2}\sec(2\alpha)+C$;

e) $C-\frac{1}{7}\csc(7\alpha)$

3) a) $\frac{2}{3}x^{3/2} - 2x + 2\sqrt{x} + C$; b) $\frac{2}{3}\left(\sqrt{x} - 1\right)^3 + C$

c) Hint: Expand $\left(\sqrt{x} - 1\right)^3$ from b), possibly by using the Binomial Theorem. Remember that our answers to a) and b) can differ by a constant term.

4) $C - \dfrac{2\left(2 + \frac{1}{\sqrt{z}}\right)^5}{5}$, though a "simplified" version may be: $C - \dfrac{2\sqrt{z}\left(2\sqrt{z} + 1\right)^5}{5z^3}$.

5) $\frac{2}{7}(u-2)^{7/2} + \frac{8}{5}(u-2)^{5/2} + \frac{10}{3}(u-2)^{3/2} + C$, or $\frac{2}{105}(u-2)^{3/2}\left(15u^2 + 24u + 67\right)$.

6) $x + \frac{1}{x+1} + C$, or $\frac{x^2 + x + 1}{x+1} + C$, or $\frac{x^2 + 2x + 2}{x+1} + C$, or $\frac{x^2}{x+1} + C$.

Hint: $\displaystyle\int \frac{x^2 + 2x}{x^2 + 2x + 1}\,dx = \int \frac{x^2 + 2x + 1 - 1}{x^2 + 2x + 1}\,dx = \int \left[1 - \frac{1}{(x+1)^2}\right] dx$.

SECTIONS 5.3/5.4: AREA and DEFINITE INTEGRALS

(Observe that the given integrands are continuous on the intervals of interest.)

1) a) $\frac{14}{3}$; b) $-\frac{14}{3}$

2) a) 36; b) 25; c) 2.5; d) $\frac{9\pi}{4}$; e) $12 + 2\pi$

3) a) 30; b) 42; c) 36; d) 36

4) $\frac{49}{4}$, or 12.25.

SECTION 5.5: PROPERTIES OF DEFINITE INTEGRALS

(Observe that the given integrands are continuous on the intervals of interest.)

1) a) 78; b) $\frac{291}{2}$ or 145.5

2) True. $1 + \sin x \geq 0$ for all real x, particularly on the interval $[0, 4\pi]$.

3) True. $-\cos^2 x \leq 0$ for all real x, particularly on the interval $[3\pi, 5\pi]$.

4) $\int_{-3}^{1} f(x)\,dx$

5) $\int_{h}^{c+h} g(t)\,dt$

6) a) 2; b) -1 (1 is technically not in the interval $(-2, 1)$.)

7) a) 6; b) 3

SECTION 5.6: FUNDAMENTAL THEOREM OF CALCULUS (FTC)

(Observe that the given integrands are continuous on the intervals of interest.)

1) a) 6; b) 5; c) $\frac{86}{3}$, or $28\frac{2}{3}$

d) $\frac{32}{5}$, or $6\frac{2}{5}$, or 6.4. The integrand is even: $\int_{-8}^{8}\left(\sqrt[3]{x^2}-2\right)dx = 2\int_{0}^{8}\left(\sqrt[3]{x^2}-2\right)dx$.

e) -8; f) 0; g) 10; h) $\sqrt{2}$

i) $\frac{17}{6}$. Hint: $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \frac{1}{2}+\frac{7}{3}$.

Observe that f is continuous on $[0,2]$. In particular, it is continuous on $[0,1]$ and on $[1,2]$.

2) a) $\frac{204{,}085}{4}$, or $51{,}021\frac{1}{4}$, or 51,021.25

b) $\frac{14}{3}$, or $4\frac{2}{3}$; c) $\frac{1}{3}$; d) $\frac{5}{36}$; e) $\frac{3}{2}\left(\sqrt{3}-1\right)\approx 1.10$; f) $1-\sqrt{2}\approx -0.41$; g) 1

3) a) and b) $\frac{727}{3}$ or $242\frac{1}{3}$. On b), we have: $\int_{-2}^{-1}(8-5x)^2\,dx = -\frac{1}{5}\int_{18}^{13}u^2\,du$.

4) $\int_0^3 |x-1|\,dx = \int_0^1\left[-(x-1)\right]dx + \int_1^3 (x-1)\,dx = \frac{5}{2}$ or 2.5, the same as in Sections 5.3/5.4, Exercise 2c.

5) 0. The integrand is odd and continuous on $[-\pi,\pi]$, which is symmetric about 0.

6) a) $\frac{1}{2}$; b) $\sqrt{3}$ ($-\sqrt{3}$ is not in the interval $(0,4)$.)

7) 0

8) $\sin^{10} x$

SECTION 5.7: NUMERICAL APPROXIMATION OF DEFINITE INTEGRALS

(Observe that the given integrands are continuous on the intervals of interest.)

1) a) 0.4728; b) 0.4637. Note: In fact, $\int_1^3 \frac{1}{1+x^2}\,dx \approx 0.4636$.

2) a) 2.2386; b) 2.3351. Note: In fact, $\int_0^{\pi}\sqrt{\sin x}\,dx \approx 2.3963$.

CHAPTER 6: APPLICATIONS OF INTEGRALS

SECTION 6.1: AREA

Assume that distances and lengths are measured in meters.

1) For parts a) and b) below, in the usual *xy*-plane …
   i) Sketch the region R bounded by the graphs of the given equations.
      Locate any intersection points of the graphs.
   ii) **Set up the integral(s)** for the area of R by integrating with respect to x.
   iii) **Set up the integral(s)** for the area of R by integrating with respect to y.
   iv) Find the area of the region by using either method from ii) or iii).
      ADDITIONAL PROBLEM: Find the area using the other method.

   a) $y = -x^2$ and $y = x^2 - 8$

   b) $y^2 = 4 - x$ and $x + 2y = 1$

2) In the *tw*-plane, sketch the regions bounded by the graphs of $w = \sin t$ and $w = \cos t$, where t is restricted to the interval $\left[0, 2\pi\right]$, and find the total area of the regions.

3) Find the area of the region bounded by the graphs of the given equations in the usual *xy*-plane. **You do not have to sketch the region.**

   a) $y = 3x^2 - 5$ and $y = x^2 + 5x - 2$

   b) $y = x\sqrt{x^2 + 16}$, $x = 0$, $x = 3$, and $y = 0$

   c) $x - y^3 = 0$ and $x + y + 2y^2 = 0$

SECTION 6.2: VOLUMES OF SOLIDS OF REVOLUTION – DISKS AND WASHERS

Assume that distances and lengths are measured in meters.
Assume that graphs are in the usual *xy*-plane, unless otherwise indicated.

1) The region R is bounded by the graphs of $x + 2y = 2$, $x = 0$, and $y = 0$.
   a) Sketch the region.
   b) Find the volume of the solid generated if R is revolved about the x-axis. First, use an integral. Then, use the formula for the volume of a cone.
   c) Find the volume of the solid generated if R is revolved about the y-axis. First, use an integral. Then, use the formula for the volume of a cone.

2) The region R is bounded by the graphs of $y = -x^2$ and $y = x^2 - 8$. You should have sketched and found the area of this region in Section 6.1, Exercise 1a.
   a) Find the volume of the solid generated if R is revolved about the x-axis.
   b) Find the volume of the solid generated if R is revolved about the y-axis.

3) The region R in the *tw*-plane is bounded by the graphs of $w = \sin t$, $w = \cos t$, $t = 0$, and $t = \dfrac{\pi}{4}$. (This region was related to Section 6.1, Exercise 2.) Find the volume of the solid generated if R is revolved about the t-axis. Hint: You will need to use a trig identity.

4) The region R in the *xy*-plane is bounded by the graphs of $y = \cos(2x)$, $y = 0$, $x = 0$, and $x = \dfrac{\pi}{4}$. Sketch R. Find the volume of the solid generated if R is revolved about the x-axis. Hint: You will need to use a trig identity.

5) The region R is bounded by the graphs of $y = 4x^2$ and $y = 2x$. Sketch R and locate any intersection points of the graphs. Find the volume of the solid generated if R is revolved about the y-axis.

6) Find the volume of a right circular cylinder of base radius r and height (or altitude) h using the Disk Method.

7) Find the volume of a right circular cone of base radius r and height (or altitude) h using the Disk Method.

8) Find the volume of a sphere of radius r using the Disk Method.

For the following problems, set up the appropriate integral(s), but do not evaluate. Your final integrals must have no general notation such as *f*, *g*, f', or *r* that can be re-expressed more precisely. You do not have to simplify your final answers.

9) The region R is bounded by the graphs of $x = 4y - y^2$ and $x = 0$. Sketch R and locate any intersection points of the graphs. **Set up the integral(s)** for the volume of the solid generated if R is revolved about the y-axis.

   ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.

10) The region R is bounded by the graphs of $y = x^2$ and $y = 4$. Sketch R and locate any intersection points of the graphs. **Set up the integral(s)** for the volume of the solid generated if R is revolved about …

    a) the line $y = 4$;
       the volume turns out to be the same as for #9 – analyze sketches to see why!
       ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.
    b) the line $y = 5$.
       ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.
    c) the line $x = 3$.
       ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.

11) The region R is bounded by the graph of $x^2 + y^2 = 1$. Sketch R. **Set up the integral(s)** for the volume of the solid generated if R is revolved about the line $x = 5$.

    ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.

    ADDITIONAL PROBLEM: Find the volume of the solid using Pappus's second centroid theorem, which states that this volume can be found by multiplying the area of the region by the distance traveled by the centroid of the region in one full revolution about the axis of revolution. (Pappus's first centroid theorem deals with surface area. See Pappus's Centroid Theorem at mathworld.wolfram.com.)
    • Note 1: In general, the centroid could lie outside of the region.
    • Note 2: We assume that the axis of revolution does not pass through the region.

SECTION 6.3: VOLUMES OF SOLIDS OF REVOLUTION – CYLINDRICAL SHELLS

Assume that distances and lengths are measured in meters.
Assume that graphs are in the usual *xy*-plane, unless otherwise indicated.

1) The region R is bounded by the graphs of $y = \tan\left(x^2\right)$, $y = 0$, $x = 0$, and $x = \dfrac{\sqrt{\pi}}{2}$.
   Set up the integral(s) for the volume of the solid generated by revolving R about the y-axis. You do not have to evaluate the integral(s). (You will in Section 7.4.)

2) The region R is bounded by the graphs of $2x - y = 12$, $x - 2y = 3$, and $x = 4$.
   Sketch R and locate any intersection points of the graphs.
   Find the volume of the solid generated by revolving R about the y-axis.

3) The region R is bounded by the graphs of $x^2 = 4y$ and $y = 4$.
   Sketch R and locate any intersection points of the graphs.
   Find the volume of the solid generated by revolving R about the x-axis.
   Try to visualize this solid.

4) The region R is bounded by the graphs of $x = y^2 + 3$ and $x - 4y = 0$.
   Sketch R and locate any intersection points of the graphs.
   Find the volume of the solid generated by revolving R about the x-axis.

MIXING METHODS

5) The region R is bounded by the graphs of $y = x^3$, $y = 8$, and $x = 0$. Sketch R. Locate any intersection points of the graphs. For each part below, **set up the integral(s)** for the volume of the solid generated by revolving R about the indicated axis. Use the indicated variable of integration; you need to figure out which method (Section 6.2 vs. Section 6.3) to use.
   a) axis is the line $x = 0$; variable of integration is x
   b) axis is the line $x = 0$; variable of integration is y
   c) axis is the line $x = 3$; variable of integration is x
   d) axis is the line $x = 3$; variable of integration is y

   ADDITIONAL PROBLEM(S): Evaluate the integrals you have set up in a), b), c), and d), and give the volumes using appropriate units. Your answers to a) and b) will be the same; your answers to c) and d) will be the same.

6) The region R is bounded by the graphs of $y = 4x^2$ and $4x + y = 8$. Sketch R. Locate any intersection points of the graphs. For each part below, **set up the integral(s)** for the volume of the solid generated by revolving R about the indicated axis. Use the indicated variable of integration; you need to figure out which method (Section 6.2 vs. Section 6.3) to use.
   a) axis is the x-axis; variable of integration is x
   b) axis is the line $x = 1$; variable of integration is x
   c) axis is the line $y = 16$; variable of integration is x

   ADDITIONAL PROBLEM(S): Evaluate the integrals you have set up in a), b), and c), and give the volumes using appropriate units.

ADDITIONAL PROBLEM

7) Use the cylindrical shells method to prove that the volume of a sphere of radius r $(r > 0)$ is $\frac{4}{3}\pi r^3$.

SECTION 6.4: VOLUMES BY CROSS SECTIONS

Assume that distances and lengths are measured in meters.
Assume that graphs are in the usual xy-plane, unless otherwise indicated.

1) Let B be the region bounded by the graphs of $x = y^2$ and $x = 9$. Sketch B. For each part below, find the volume of the solid that has B as its base if every cross section by a plane perpendicular to the x-axis is …
   a) … a square
   b) … a semicircle with diameter lying on B
   c) … an equilateral triangle

2) Let B be the region bounded by the graph of $x^2 + y^2 = a^2$, where a is a positive constant. Sketch B. Find the volume of the solid that has B as its base if every cross section by a plane perpendicular to the x-axis is a square.

3) Let B be the region bounded by the graphs of $y = x^2$ and $y = 4$. Sketch B. Find the volume of the solid that has B as its base if every cross section by a plane perpendicular to the x-axis is an isosceles right triangle with hypotenuse on B.

SECTION 6.5: ARC LENGTH and SURFACES OF REVOLUTION

Assume that distances and lengths are measured in meters.
Assume that graphs are in the usual *xy*-plane, unless otherwise indicated.

1) Consider the portion of the graph of $y = x^3 + 1$ from the point $A(1, 2)$ to the point $B(3, 28)$.
   a) **Set up the integral(s)** for the arc length of the portion by integrating with respect to x.
   b) **Set up the integral(s)** for the arc length of the portion by integrating with respect to y.
   c) **Set up the integral(s)** for the area of the surface of revolution that is obtained when the portion is revolved about the x-axis.
   d) **Set up the integral(s)** for the area of the surface of revolution that is obtained when the portion is revolved about the y-axis.

2) **Set up the integral(s)** for the area of the surface of revolution that is obtained when the portion of the graph of $y^2 = x$ from $A(0, 0)$ to $B(4, 2)$ is revolved about the …
   a) x-axis. Also, evaluate the integral(s). Give the exact surface area and also an approximation rounded off to four significant digits.
   b) y-axis. Do not evaluate the integral(s).

3) Find the surface area of a sphere of radius r.
   • Technical Note: Simplify the integrand before writing your first integral. There is technically a division-by-zero issue that arises when, say, $x = r$ and/or $x = -r$, which would lead to an "improper integral"; ignore this for now and proceed.

ADDITIONAL PROBLEM

4) Find the lateral surface area (which excludes the base area) of a right circular cone of base radius r and height (or altitude) h using …
   a) the methods of this section.
   b) Pappus's first centroid theorem, which states that this surface area can be found by multiplying the length of the arc being revolved by the distance traveled by the centroid of the arc in one full revolution about the axis of revolution. (See Pappus's Centroid Theorem at mathworld.wolfram.com.)

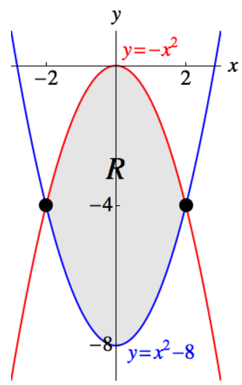
CHAPTER 6: APPLICATIONS OF INTEGRALS

SECTION 6.1: AREA

1)

a)

i)



ii) $\int_{-2}^{2}\left[\left(-x^2\right)-\left(x^2-8\right)\right]dx$, or $2\int_{0}^{2}\left[\left(-x^2\right)-\left(x^2-8\right)\right]dx$, or

$4\int_{0}^{2}\left[\left(-x^2\right)-\left(-4\right)\right]dx$ (by symmetry)

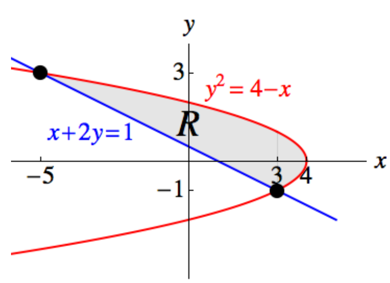
iii) $\int_{-8}^{-4}2\sqrt{y+8}\,dy + \int_{-4}^{0}2\sqrt{-y}\,dy$, or $2\int_{-8}^{-4}\sqrt{y+8}\,dy + 2\int_{-4}^{0}\sqrt{-y}\,dy$,

or $4\int_{-4}^{0}\sqrt{-y}\,dy$ (by symmetry)

iv) $\dfrac{64}{3}\text{ m}^2$, or $21\dfrac{1}{3}\text{ m}^2$

b)

i)

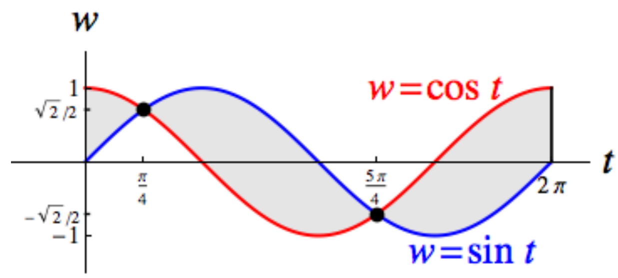


ii) $\int_{-5}^{3}\left(\sqrt{4-x}-\dfrac{1-x}{2}\right)dx + \int_{3}^{4}2\sqrt{4-x}\,dx$

iii) $\int_{-1}^{3}\left[\left(4-y^2\right)-\left(1-2y\right)\right]dy$

iv) $\dfrac{32}{3}\text{ m}^2$

2)



$4\sqrt{2}$ m^2. Hint: The setup is given by:

$\int_0^{\pi/4} (\cos t - \sin t)\, dt \; + \; \int_{\pi/4}^{5\pi/4} (\sin t - \cos t)\, dt \; + \; \int_{5\pi/4}^{2\pi} (\cos t - \sin t)\, dt$.

3)

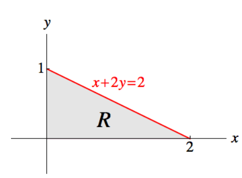
a) $\dfrac{343}{24}$ m^2, or $14\dfrac{7}{24}$ m^2. Hint: Setup is: $\int_{-1/2}^{3} \left[\left(x^2 + 5x - 2 \right) - \left(3x^2 - 5 \right) \right] dx$.

b) $\dfrac{61}{3}$ m^2, or $20\dfrac{1}{3}$ m^2. Hint: Setup is: $\int_0^3 x\sqrt{x^2+16}\, dx$.

c) $\dfrac{1}{12}$ m^2. Hint: Setup is: $\int_{-1}^{0} \left[\left(-y - 2y^2 \right) - \left(y^3 \right) \right] dy$.

SECTION 6.2: VOLUMES OF SOLIDS OF REVOLUTION – DISKS AND WASHERS

1) a)



b) $\dfrac{2\pi}{3}$ m^3. Hint: Setup is: $\int_0^2 \pi \left(\dfrac{2-x}{2} \right)^2 dx$.

c) $\dfrac{4\pi}{3}$ m^3. Hint: Setup is: $\int_0^1 \pi (2-2y)^2\, dy$.

2) a) $\dfrac{512\pi}{3}$ m^3. Hint: Setup is: $\int_{-2}^{2} \left[\pi \left(0 - \left(x^2 - 8 \right) \right)^2 - \pi \left(0 - \left(-x^2 \right) \right)^2 \right] dx$, or

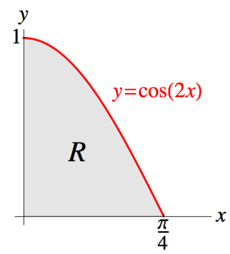
$2\int_0^2 \left[\pi \left(0 - \left(x^2 - 8 \right) \right)^2 - \pi \left(0 - \left(-x^2 \right) \right)^2 \right] dx$ by symmetry.

b) 16π m^3. Hint 1: Setup is: $\int_{-8}^{-4} \pi (y+8)\, dy \; + \; \int_{-4}^{0} \pi (-y)\, dy$, or

$2\int_{-4}^{0} \pi (-y)\, dy$ by symmetry. Hint 2: You only want to revolve half of the region 360° around the axis of revolution.

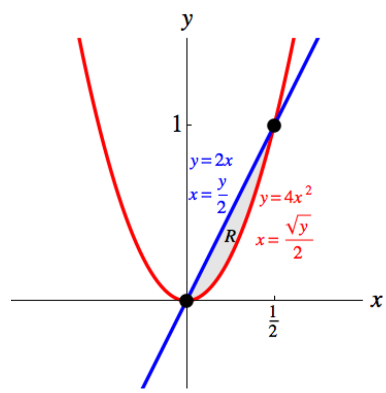
3) $\frac{\pi}{2}$ m^3. Hint: Setup is: $\int_0^{\pi/4} \left[\pi(\cos t)^2 - \pi(\sin t)^2 \right] dt$.

4)



$\frac{\pi^2}{8}$ m^3. Hint: Setup is: $\int_0^{\pi/4} \pi \left[\cos(2x) \right]^2 dx = \pi \int_0^{\pi/4} \frac{1+\cos(4x)}{2} dx$.

5)



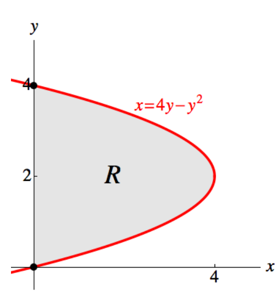
$\frac{\pi}{24}$ m^3. Hint: Setup is: $\int_0^1 \left[\pi\left(\frac{\sqrt{y}}{2}\right)^2 - \pi\left(\frac{y}{2}\right)^2 \right] dy$.

6) $\pi r^2 h$ (in cubic meters). Hint: Possible setup is: $\int_0^h \pi r^2 \, dx$.

7) $\frac{1}{3}\pi r^2 h$ (in cubic meters). Hint: Possible setup is: $\int_0^h \pi \left(\frac{r}{h}x\right)^2 dx$.

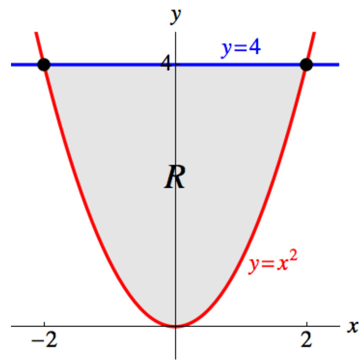
8) $\frac{4}{3}\pi r^3$ (in cubic meters). Hint: Possible setup is: $2\int_0^r \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$.

9)



$\int_0^4 \pi\left(4y - y^2\right)^2 dy$ (in cubic meters). Additional Problem: The volume is $\frac{512\pi}{15}$ m^3.

10)



a) $\int_{-2}^{2} \pi\left(4-x^2\right)^2\,dx$ (in cubic meters), or $2\int_{0}^{2} \pi\left(4-x^2\right)^2\,dx$ (in cubic meters) by exploiting symmetry. Additional Problem: The volume is $\dfrac{512\pi}{15}$ m^3. (Why do we get the same answer as we did for #9? Draw graphs and see!)

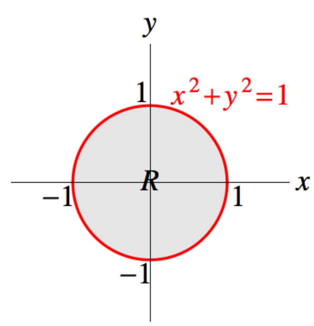
b) $\int_{-2}^{2} \left[\pi\left(5-x^2\right)^2 - \pi(1)^2\right] dx$ (in cubic meters), or $2\int_{0}^{2} \left[\pi\left(5-x^2\right)^2 - \pi(1)^2\right] dx$ (in cubic meters) by exploiting symmetry.

Additional Problem: The volume is $\dfrac{832\pi}{15}$ m^3.

c) $\int_{0}^{4} \left[\pi\left(3-\left[-\sqrt{y}\right]\right)^2 - \pi\left(3-\sqrt{y}\right)^2\right] dy$ (in cubic meters).

Additional Problem: The volume is 64π m^3.

11)



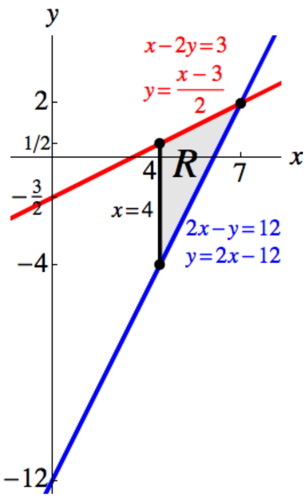
$\int_{-1}^{1} \left[\pi\left(5-\left[-\sqrt{1-y^2}\right]\right)^2 - \pi\left(5-\sqrt{1-y^2}\right)^2\right] dy$ (in cubic meters), or

$2\int_{0}^{1} \left[\pi\left(5-\left[-\sqrt{1-y^2}\right]\right)^2 - \pi\left(5-\sqrt{1-y^2}\right)^2\right] dy$ (in cubic meters) by exploiting symmetry. Additional Problems: The volume is $10\pi^2$ m^3. Geometry may help!

SECTION 6.3: VOLUMES OF SOLIDS OF REVOLUTION – CYLINDRICAL SHELLS

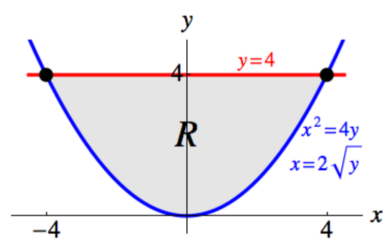
1) $\int_0^{\sqrt{\pi}/2} 2\pi x \tan\left(x^2\right) dx$ (in cubic meters). Observe that, as x varies from 0 to $\dfrac{\sqrt{\pi}}{2}$, x^2 varies from 0 to $\dfrac{\pi}{4}$, so $\tan\left(x^2\right) \geq 0$ on the interval $\left[0, \dfrac{\sqrt{\pi}}{2}\right]$.

2)



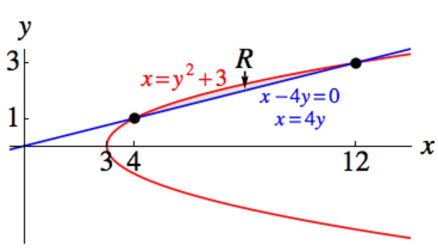
$\dfrac{135\pi}{2}$ m^3. Hint: Setup is: $\int_4^7 2\pi x\left[\dfrac{x-3}{2}-\left(2x-12\right)\right]dx$.

3)



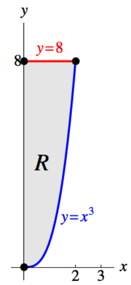
$\dfrac{512\pi}{5}$ m^3. Hint: Setup is: $2\int_0^4 2\pi y\left(2\sqrt{y}\right)dy$ by symmetry. Visualize the solid: Imagine packing foam. The solid corresponds to the space between a “squished convex” hourglass and a cylinder in which it fits snugly.

4)



$\dfrac{16\pi}{3}$ m^3. Hint: Setup is: $\int_1^3 2\pi y\left[4y-\left(y^2+3\right)\right]dy$.

5)



a) $\int_0^2 2\pi x\left(8-x^3\right)dx$ (in cubic meters). Hint: Use cylinders / cylindrical shells.

Additional Problem: The volume is $\dfrac{96\pi}{5}$ m^3.

b) $\int_0^8 \pi\left(\sqrt[3]{y^2}\right)dy$ (in cubic meters). Hint: Use disks.

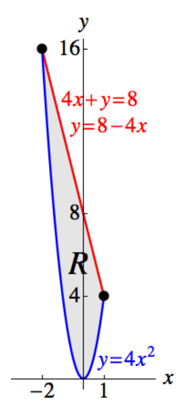
Additional Problem: The volume is $\dfrac{96\pi}{5}$ m^3, same as for a).

c) $\int_0^2 2\pi\left(3-x\right)\left(8-x^3\right)dx$ (in cubic meters). Hint: Use cylinders / cylindrical shells. Additional Problem: The volume is $\dfrac{264\pi}{5}$ m^3.

d) $\int_0^8 \left[\pi\left(3\right)^2-\pi\left(3-\sqrt[3]{y}\right)^2\right]dy$ (in cubic meters). Hint: Use washers.

Additional Problem: The volume is $\dfrac{264\pi}{5}$ m^3, same as for c).

6)



a) $\int_{-2}^1 \left[\pi\left(8-4x\right)^2-\pi\left(4x^2\right)^2\right]dx$ (in cubic meters). Hint: Use washers.

Additional Problem: The volume is $\dfrac{1152\pi}{5}$ m^3.

b) $\int_{-2}^1 2\pi\left(1-x\right)\left(\left[8-4x\right]-\left[4x^2\right]\right)dx$ (in cubic meters). Hint: Use cylinders / cylindrical shells. Additional Problem: The volume is 54π m^3.

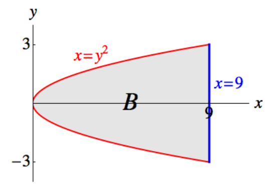
c) $\int_{-2}^{1}\left[\pi\left(16-4x^2\right)^2-\pi\left(16-\left[8-4x\right]\right)^2\right]dx$ (in cubic meters).

Hint: Use washers. Additional Problem: The volume is $\frac{1728\pi}{5}\text{ m}^3$.

7) Hint: Setup is: $2\int_0^r 2\pi x\sqrt{r^2-x^2}\,dx$.

SECTION 6.4: VOLUMES BY CROSS SECTIONS

1) Sketch of B:

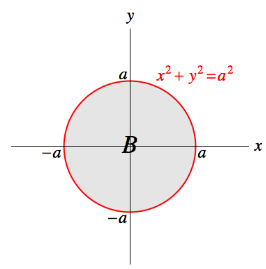


a) 162 m^3. Hint: Setup is: $\int_0^9\left(2\sqrt{x}\right)^2dx$.

b) $\frac{81\pi}{4}\text{ m}^3$. Hint: Setup is: $\int_0^9\frac{1}{2}\pi\left(\sqrt{x}\right)^2dx$.

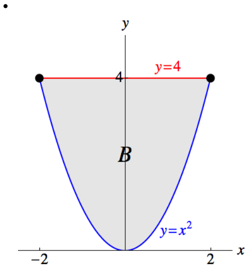
c) $\frac{81\sqrt{3}}{2}\text{ m}^3$. Hint: Setup is: $\int_0^9\sqrt{3}x\,dx$.

2) Sketch of B:



$\frac{16a^3}{3}\text{ m}^3$. Hint: Setup is: $2\int_0^a\left(2\sqrt{a^2-x^2}\right)^2dx$.

3) Sketch of B:



$\frac{128}{15}\text{ m}^3$, or $8\frac{8}{15}\text{ m}^3$.

Hint: Setup is: $2\int_0^2\frac{1}{2}\left[\frac{1}{\sqrt{2}}\left(4-x^2\right)\right]^2dx$, or $2\int_0^2\frac{1}{2}\cdot\frac{1}{2}\left(4-x^2\right)^2dx$.

SECTION 6.5: ARC LENGTH and SURFACES OF REVOLUTION

Note: Observe that the integrands are continuous on the closed intervals of interest.

1)

a) $\int_1^3 \sqrt{1+\left(3x^2\right)^2}\, dx$ (in meters)

b) $\int_2^{28} \sqrt{1+\left[\frac{1}{3(y-1)^{2/3}}\right]^2}\, dy$ (in meters)

c) $\int_1^3 2\pi\left(x^3+1\right)\sqrt{1+\left(3x^2\right)^2}\, dx$ (in square meters)

d) $\int_2^{28} 2\pi(y-1)^{1/3}\sqrt{1+\left[\frac{1}{3(y-1)^{2/3}}\right]^2}\, dy$ (in square meters)

2)

a) $\int_0^4 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\, dx = \frac{\pi}{6}\left[17\sqrt{17}-1\right]\text{ m}^2 \approx 36.18\text{ m}^2$;

b) $\int_0^2 2\pi y^2\sqrt{1+4y^2}\, dy$ (in square meters)

3) $4\pi r^2$ (in square meters). Hint: Setup is: $2\int_0^r 2\pi\sqrt{r^2-x^2}\sqrt{1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2}\, dx$.

Note: The above setup leads to the integral $\int_0^r 4\pi r\, dx$, which has a constant integrand. This implies that, on a fine regular partition (imagine forcing an unhusked coconut through a shredder), the corresponding pieces of the sphere have approximately equal surface areas. Although the "average radii" of these pieces are shrinking as, say, $x \to r^-$, the pieces are also slanting more steeply. (If you eat the shredded coconut pieces, the "end pieces" will be about as filling as the "middle pieces.")

4) a) and b). $\pi r\sqrt{r^2+h^2}$ (in square meters). This can be thought of as $\pi r l$ (in square meters), where $l=\sqrt{r^2+h^2}$, the slant height of the cone.

Hint on a): Evaluate $\int_0^h 2\pi\left(\frac{r}{h}x\right)\sqrt{1+\left(\frac{r}{h}\right)^2}\, dx$.

CHAPTER 7: LOGARITHMIC and EXPONENTIAL FUNCTIONS

SECTION 7.1: INVERSE FUNCTIONS

1) Let $f(x) = 3x + 4$.

   a) What is the slope of the line with equation $y = f(x)$?

   b) Find $f^{-1}(x)$, the rule for the inverse function of f.
   You may want to review Section 1.9 on Inverse Functions in the Precalculus notes.

   c) What is the slope of the line with equation $y = f^{-1}(x)$?
   Compare this slope with the slope from part a).

2) Let $f(x) = x^3$. Observe that $f(2) = 8$.

   a) Find $f'(2)$.

   b) Let $g(x) = f^{-1}(x)$, the rule for the inverse function of f. Find $g(x)$.

   c) Find $g'(8)$. Compare this with your answer from part a).

SECTION 7.2: ln x

1) Find the following derivatives. Simplify where appropriate.
Do not leave negative exponents in your final answer.
You do not have to simplify radicals or rationalize denominators.

   a) Let $f(x) = \ln(5x^3 - x + 1)$. Find $f'(x)$.

   b) Find $\dfrac{d}{dx}\left[\ln(x^3 + x^2)\right]$.

   c) Find $D_x\left[\ln|3x + 7|\right]$.

   d) Let $g(t) = \ln\left(|7 - 4t|^{10}\right)$. Find $g'(t)$.

   e) Let $y = \ln(x^3) + (\ln x)^3$. Find $\dfrac{dy}{dx}$.

   f) Find $\dfrac{d}{dw}\left[\dfrac{w^2 \ln w}{\ln\left(\dfrac{1}{w}\right)}\right]$. Hint: Simplify first!

   g) Find $\dfrac{d}{dw}\left[\dfrac{w^2 \ln w}{1 + \ln w}\right]$.

2) Let $y = \ln\left[\dfrac{(x^4 + 1)^3\left(\sqrt{x}\right)}{(3x - 4)^5}\right]$. Find $\dfrac{dy}{dx}$.

   Before performing any differentiation, apply appropriate laws of logarithms wherever they apply. You do not have to write your final answer as a single fraction.

3) Find $D_x\left(\ln|\sec x|\right)$. Based on your result, write the corresponding indefinite integral statement. We will discuss this further in Section 7.4.

4) We will find $D_\theta\left(\cos^7 \theta\right)$ in two different ways.

   a) Apply the Generalized Power Rule of Differentiation directly.
   b) Use Logarithmic Differentiation. Apply appropriate laws of logarithms wherever they apply. Observe that your answers to a) and b) should be equivalent, at least where $\cos\theta \neq 0$.

5) We will find $D_x\left[\left(3x^2+2\right)^4\left(\sqrt{3x+5}\right)\right]$ in two different ways.

   a) Apply the Product Rule and the Generalized Power Rule of Differentiation directly. Simplify completely, and write your final answer as a single non-compound fraction. Do not leave negative exponents in your final answer.
   b) Use Logarithmic Differentiation. Apply appropriate laws of logarithms wherever they apply. For now, you do not have to write your final answer as a single fraction.
   c) Simplify your answer to part b) completely, and write your final answer as a single non-compound fraction. Compare with your answer to part a).

6) Let $f(x)=\ln(\ln x)$. Consider the graph of f in the usual xy-plane.

   a) What is the domain of f?
   b) Find $f'(x)$.
   c) Find $f''(x)$.
   d) What do f' and f'' tell us about the graph of f?
   e) Find an equation of the tangent line to the graph of f at the point $\left(e^2, \ln 2\right)$.

7) Use derivatives to explain why x increases (or "grows") faster than $\ln x$ does as x increases through the interval $(1, \infty)$.

8) Use Logarithmic Differentiation to prove the Power Rule of Differentiation: $D_x\left(x^n\right)=nx^{n-1}$, where n is an arbitrary real number. Assume $x \neq 0$.

SECTION 7.3: e^x

1) Find the following derivatives. Simplify where appropriate.
   Do not leave negative constant exponents in your final answer.
   You do not have to simplify radicals or rationalize denominators.
   a) Let $f(x) = e^{-8x}$. Find $f'(x)$.
   b) Find $\dfrac{d}{dx}\left(\sqrt{1+2x+3e^{4x}}\right)$.
   c) Find $D_x\left(e^{4x^3+x}\right)$.
   d) Let $g(t) = e^{e^{e^t}}$. Find $g'(t)$.
   e) Let $y = \dfrac{x^2 e^x}{e^x + 1}$. Find $\dfrac{dy}{dx}$.
   f) Find $D_x\left(\dfrac{1}{e^x} + e^{\frac{1}{x}}\right)$.
   g) Find $D_x\left(e^{x \ln x}\right)$.
   h) Find $\dfrac{d}{d\theta}\left[e^{\ln(\sin\theta)} + \ln\left(e^{\cos\theta}\right)\right]$, where $\sin\theta > 0$. Hint: Simplify first!
   i) Find $D_x\left[\sec\left(e^x\right)\right]$.
   j) Let $h(r) = \tan^5\left(4e^{6r}\right)$. Find $h'(r)$.
   k) Find $\dfrac{d}{dx}\left[e^{3\csc(2x)+1}\right]$.
   l) Let $f(\theta) = \ln\left[\sin\left(e^{-\theta}\right)\right]$. Find $f'(\theta)$.
   m) Let $g(x) = e^{4x}\cot\left(\sqrt{x}\right)$. Find $g'(x)$.
   n) Find $D_x\left(e^{\pi}\right)$.

2) Consider the given equation $\ln\left(x^2 y\right)+2y^6 - x^2 = 7 + e^6$, where $x > 0$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.

a) Find $\frac{dy}{dx}$, also known as y'.

b) Verify that the point $P\left(e^3, e\right)$ lies on the graph of the given equation.

c) Evaluate $\left[\frac{dy}{dx}\right]_{\left(e^3,\, e\right)}$.

3) Consider the given equation $e^{xy} = \sec y$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$. Find $\frac{dy}{dx}$, also known as y'.

4) (Radioactive Decay). The amount remaining (in grams) of a radioactive substance t minutes after noon is given by $f(t) = ae^{-bt}$, where a and b are positive real constants. Show that the rate of decay of the substance is directly proportional to the amount of the substance that remains. (Note: a is the "initial" amount of the substance remaining at $t = 0$.) You will come back to these ideas when you study Differential Equations.

5) **ADDITIONAL PROBLEM:** (Statistics). The standard normal probability density function in statistics is given by: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Sketch the graph of $y = f(x)$ in the usual *xy*-plane.

- Find the domain of *f*.
- Comment on the signs of values of *f*.
- State whether f is even, odd, or neither, and incorporate any corresponding symmetry in your graph.
- Find and indicate on your graph any horizontal asymptotes (HAs), and justify them using limits.
- Locate any local maximum points, local minimum points, and inflection points.
- Find the intervals on which f is increasing / decreasing, and have your graph show that.
- Find the *x*-intervals on which the graph of $y = f(x)$ is concave up / concave down, and have your graph show that.

SECTION 7.4: INTEGRATION and LOG / EXP. FUNCTIONS

1) Evaluate the following indefinite integrals. You may use C, D, etc. as representing arbitrary constants.

   a) $\int \frac{1}{2x-3}\, dx$. Try using Guess-and-Check here.

   b) $\int \left(e^{7x} + \frac{1}{e^{7x}} \right) dx$. Try rewriting and using Guess-and-Check here.

   c) $\int \tan(3x)\, dx$

   d) $\int \cot\left(\frac{x}{5}\right) dx$

   e) $\int \frac{5x}{4x^2+3}\, dx$

   f) $\int x^2 e^{5x^3}\, dx$

   g) $\int 6\theta \sec\left(\theta^2 + e\right) d\theta$

   h) $\int \frac{x-2}{x^2-4x+5}\, dx$

   i) $\int \frac{(t+3)^2}{t}\, dt$

   j) $\int (\csc x + 4)^2\, dx$

   k) $\int \frac{\sin(\ln x)}{x}\, dx$

   l) $\int \frac{\pi e^{\sqrt{x}}}{7\sqrt{x}}\, dx$

   m) $\int \frac{\sec^2 x + 1}{x + \tan x}\, dx$

   n) $\int \frac{(e^x+1)^2}{e^x}\, dx$

   o) $\int \frac{e^x + e^{-x}}{(e^x - e^{-x})^2}\, dx$

p) $\int \frac{e^x}{e^x + 1}\, dx$

q) $\int \frac{e^x}{\cos\left(3e^x - e\right)}\, dx$

r) $\int \frac{e^{\sin x}}{\sec x}\, dx$

s) $\int \frac{\cos^2 \theta}{\sin\theta}\, d\theta$

t) **ADDITIONAL PROBLEM:** $\int \frac{\csc\left(e^{-2x}\right)\cot\left(e^{-2x}\right)}{e^{2x}\left[1+\csc\left(e^{-2x}\right)\right]}\, dx$

2) We will consider the definite integral $\int_1^2 \frac{1}{x}\, dx$.

a) Use the Trapezoidal Rule to approximate $\int_1^2 \frac{1}{x}\, dx$. Use a regular partition with $n = 4$ subintervals. Round off your answer to three significant digits.

b) Evaluate $\int_1^2 \frac{1}{x}\, dx$ using the Fundamental Theorem of Calculus. Give an exact answer, and also approximate it to three significant digits. Compare with part a).

3) Show that $\int \ln x\, dx = x\ln x - x + C$ by showing that $D_x\left(x\ln x - x\right) = \ln x$. In Chapter 9, you will use Integration by Parts to perform the integration directly.

4) **ADDITIONAL PROBLEM:** $\int \csc x\, dx = \ln\left|\csc x - \cot x\right| + C$. Alternatively, $\int \csc x\, dx = -\ln\left|\csc x + \cot x\right| + C$. Show that $-\ln\left|\csc x + \cot x\right| = \ln\left|\csc x - \cot x\right|$.

5) (Differential equations and motion). The velocity of a particle traveling along a coordinate line is given by $v(t) = 4e^{2t} + 3e^{-2t}$ in feet per second, where t is measured in seconds. We are given the initial condition $s(0) = 4$ feet, meaning that the position of the particle at time $t = 0$ is 4 feet along the line. Find the position function rule $s(t)$ for the particle.

6) (Volume of a solid of revolution). The region R in the usual xy-plane is bounded by the graphs of $y = e^{-x^2}$, $x = 0$, $x = 1$, and $y = 0$. Find the volume of the solid generated if R is revolved about the y-axis. Give an exact answer and also an approximate answer rounded off to four significant digits. (Assume that distances and lengths are measured in meters.)

7) (Volume of a solid of revolution). In Section 6.3, Exercise 1, you should have found that the volume of the indicated solid was given by: $\int_0^{\sqrt{\pi/2}} 2\pi x \tan\left(x^2\right) dx$. Evaluate this integral, and give the volume using appropriate units. Give an exact answer and also an approximate answer rounded off to four significant digits. (Assume that distances and lengths are measured in meters.)

SECTION 7.5: BEYOND *e* – NONNATURAL BASES

1) Find the following derivatives. Simplify where appropriate.
Do not leave negative constant exponents in your final answer.
You do not have to simplify radicals or rationalize denominators.
   a) Find $D_x\left(x^e + e^e + e^x + 2^x - 2^{3x^4+x}\right)$.
   b) Find $D_x\left(\ln x + \log_2 x\right)$.
   c) Let $f(x) = \log\left[\left|x^7 - 4x^3 + 2\right|^4\right]$. Find $f'(x)$.
   d) Find $\frac{d}{dx}\left(x^\pi \pi^x\right)$.
   e) Let $g(t) = 3^{\sec(5t)}$. Find $g'(t)$.
   f) Let $h(r) = \log_6(\ln r)$. Find $h'(r)$.
   g) Find $D_x\left[(x+3)^{x^2}\right]$. Try to simplify your final answer.
   h) Find $D_x\left(x^{\tan x}\right)$. Try to simplify your final answer.

2) (Revisiting Exercise 1h). An alternative to Logarithmic Differentiation may be performed as follows. Disregarding domain issues, observe that:
$x^{\tan x} = e^{\ln\left(x^{\tan x}\right)} = e^{(\tan x)(\ln x)}$. Find $D_x\left(x^{\tan x}\right)$ by finding $D_x\left[e^{(\tan x)(\ln x)}\right]$.

3) Evaluate the following indefinite integrals. You may use C, D, etc. as representing arbitrary constants.
   a) $\int\left(x^e + e^e + e^x + 2^x - 9\pi^x\right) dx$; b) $\int 7^{5x+3}\, dx$; c) $\int \frac{1}{x\log x}\, dx$

4) Evaluate $\int_0^1 \frac{10^x}{10^x + 1}\, dx$.

CHAPTER 7: LOGARITHMIC and EXPONENTIAL FUNCTIONS

SECTION 7.1: INVERSE FUNCTIONS

1) a) 3; b) $f^{-1}(x)=\dfrac{x-4}{3}$, or $\dfrac{1}{3}x-\dfrac{4}{3}$; c) $\dfrac{1}{3}$, which is the reciprocal of 3

2) a) 12; b) $g(x)=\sqrt[3]{x}$, or $x^{1/3}$; c) $\dfrac{1}{12}$, which is the reciprocal of 12

SECTION 7.2: ln *x*

1)

a) $\dfrac{15x^2-1}{5x^3-x+1}$

b) $\dfrac{3x+2}{x^2+x}$, or $\dfrac{3x+2}{x(x+1)}$. (Remember to simplify!)

c) $\dfrac{3}{3x+7}$

d) $\dfrac{-40}{7-4t}$, or $\dfrac{40}{4t-7}$. Hint: Use the Power Rule of Logarithms first.

e) $\dfrac{3+3(\ln x)^2}{x}$, or $\dfrac{3\left[1+(\ln x)^2\right]}{x}$.
Hint: Use the Power Rule of Logarithms on the first term.

f) $-2w$. Hint: $\ln\left(\dfrac{1}{w}\right)=\ln\left(w^{-1}\right)=-\ln w$.

g) $$\frac{(1+\ln w)(2w\ln w+w)-\left(w^2\ln w\right)\left(\frac{1}{w}\right)}{(1+\ln w)^2}=\frac{w\left[1+2\ln w+2(\ln w)^2\right]}{(1+\ln w)^2}$$

2) $\dfrac{12x^3}{x^4+1}+\dfrac{1}{2x}-\dfrac{15}{3x-4}$, or $\dfrac{12x^3}{x^4+1}+\dfrac{1}{2x}+\dfrac{15}{4-3x}$, or $\dfrac{45x^5-100x^4-27x-4}{2x\left(x^4+1\right)(3x-4)}$.

Hint: $\ln\left[\dfrac{\left(x^4+1\right)^3\left(\sqrt{x}\right)}{(3x-4)^5}\right]=3\ln\left(x^4+1\right)+\dfrac{1}{2}\ln x-5\ln(3x-4)$ by laws of logarithms.

3) $\tan x$. We then (finally) have: $\int \tan x \, dx = \ln\left| \sec x \right| + C$.

4) a) and b) $-7\sin\theta\cos^6\theta$. Note: You may have obtained $-7\tan\theta\cos^7\theta$ for part b). This is equivalent to $-7\sin\theta\cos^6\theta$, where $\cos\theta \neq 0$. Logarithmic Differentiation does not apply for values of θ that make $\cos\theta = 0$ in this problem.

5)

a) $\dfrac{3\left(51x^2+80x+2\right)\left(3x^2+2\right)^3}{2\sqrt{3x+5}}$; b) $\left[\dfrac{24x}{3x^2+2} + \dfrac{3}{2\left(3x+5\right)}\right]\left(3x^2+2\right)^4\left(\sqrt{3x+5}\right)$;

c) Your answer should effectively be the same as your answer to part a).

6)

a) $\left(1, \infty\right)$. Hint: We require $x > 0$ and $\ln x > 0$.

b) $\dfrac{1}{x\ln x}$

c) $-\dfrac{1+\ln x}{x^2\left(\ln x\right)^2}$, or $-\dfrac{1+\ln x}{\left(x\ln x\right)^2}$

d) On $\mathrm{Dom}\left(f\right) = \left(1, \infty\right)$, $f'\left(x\right) > 0$ and $f''\left(x\right) < 0$. Therefore, f is increasing and the graph of f is concave down on the x-interval $\left(1, \infty\right)$.

e) Point-Slope Form: $y - \ln 2 = \dfrac{1}{2e^2}\left(x - e^2\right)$,

Slope-Intercept Form: $y = \dfrac{1}{2e^2}x + \left(\ln 2 - \dfrac{1}{2}\right)$, or $y = \dfrac{1}{2e^2}x + \dfrac{2\ln 2 - 1}{2}$

7) • $D_x\left(x\right) = 1$, and $D_x\left(\ln x\right) = \dfrac{1}{x} < 1$ whenever $x > 1$.

• Note that $1 > 0$, and also $\dfrac{1}{x} > 0$ whenever $x > 1$; therefore, x and $\ln x$ are increasing with respect to x on the interval $\left(1, \infty\right)$.

• Alternately, because $D_x\left(x - \ln x\right) = 1 - \dfrac{1}{x} > 0$ whenever $x > 1$, we can conclude that the "gap" $x - \ln x$ is increasing on the interval $\left(1, \infty\right)$, and therefore x is increasing faster than $\ln x$ is.

8) Hints: Let $y = x^n$. Apply Implicit Differentiation to both sides of $\ln y = \ln\left(x^n\right)$.

SECTION 7.3: e^x

1)

a) $-8e^{-8x}$, or $-\dfrac{8}{e^{8x}}$

b) $\dfrac{1+6e^{4x}}{\sqrt{1+2x+3e^{4x}}}$, or $\dfrac{6e^{4x}+1}{\sqrt{3e^{4x}+2x+1}}$

c) $\left(12x^2+1\right)e^{4x^3+x}$

d) e^{e^t+t}. Hint: $e^t e^{e^t}$ simplifies to this.

e) $\dfrac{xe^x\left(2e^x+x+2\right)}{\left(e^x+1\right)^2}$. Hint: $\dfrac{\left(e^x+1\right)\left(2xe^x+x^2e^x\right)-\left(x^2e^x\right)\left(e^x\right)}{\left(e^x+1\right)^2}$ simplifies to this.

f) $-\dfrac{e^{\frac{1}{x}}}{x^2}-e^{-x}$, or $-\dfrac{e^{\frac{1}{x}}}{x^2}-\dfrac{1}{e^x}$, or $-\dfrac{e^{x+\frac{1}{x}}+x^2}{x^2e^x}$, or $-\dfrac{e^{\frac{x^2+1}{x}}+x^2}{x^2e^x}$

g) $e^{x\ln x}\left(\ln x+1\right)$, or $x^x\left(\ln x+1\right)$

h) $\cos\theta-\sin\theta$

i) $e^x\sec\left(e^x\right)\tan\left(e^x\right)$

j) $120e^{6r}\sec^2\left(4e^{6r}\right)\tan^4\left(4e^{6r}\right)$

k) $-6\csc(2x)\cot(2x)e^{3\csc(2x)+1}$

l) $-e^{-\theta}\cot\left(e^{-\theta}\right)$, or $-\dfrac{\cot\left(\dfrac{1}{e^\theta}\right)}{e^\theta}$

m) $4e^{4x}\cot\left(\sqrt{x}\right)-\dfrac{e^{4x}\csc^2\left(\sqrt{x}\right)}{2\sqrt{x}}$, or $\dfrac{e^{4x}\left[8\left(\sqrt{x}\right)\cot\left(\sqrt{x}\right)-\csc^2\left(\sqrt{x}\right)\right]}{2\sqrt{x}}$

n) 0

2)

a) $\dfrac{2y\left(x^2-1\right)}{x\left(12y^6+1\right)}$, or $\dfrac{2x^2y-2y}{12xy^6+x}$; b) $\ln\left[\left(e^3\right)^2 e\right]+2e^6-\left(e^3\right)^2=7+e^6$

c) $\dfrac{2\left(e^6-1\right)}{e^2\left(1+12e^6\right)}$, or $\dfrac{2e^6-2}{e^2+12e^8}$

3) $-\dfrac{ye^{xy}}{xe^{xy}-\sec y\tan y}$, or $\dfrac{ye^{xy}}{\sec y\tan y-xe^{xy}}$, or $\dfrac{y}{\tan y-x}$ (because $\sec y=e^{xy}$; this is more easily seen if we had taken the natural logarithm ("ln") of both sides)

4) Hint: The rate of change of f with respect to t is given by $f'(t)$. The rate of decay is given by $-f'(t)$. Show that $-f'(t)$ is equal to a positive real constant times $f(t)$. In particular, $-f'(t)=b\left(ae^{-bt}\right)=b\cdot f(t)$.

5)

$\mathrm{Dom}(f)=(-\infty,\infty)$.

$f(x)>0$ for all real values of x. Observe that $e^{-\frac{x^2}{2}}=\dfrac{1}{e^{\frac{x^2}{2}}}>0$ for all real x.

f is even, so its graph is symmetric about the y-axis.

HA: only $y=0$, because $\lim\limits_{x\to\infty} f(x)=0$, and $\lim\limits_{x\to-\infty} f(x)=0$.

$f'(x)=-\dfrac{1}{\sqrt{2\pi}}xe^{-\frac{x^2}{2}}$. Observe that $e^{-\frac{x^2}{2}}>0$ for all real values of x.

CN: 0. Point at critical number: $\left(0,\dfrac{1}{\sqrt{2\pi}}\right)$, a local maximum point.

f is increasing on $(-\infty,0]$.

f is decreasing on $[0,\infty)$.

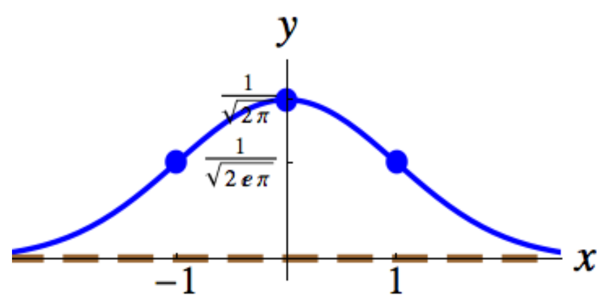
$f''(x)=\dfrac{1}{\sqrt{2\pi}}\left(x^2-1\right)e^{-\frac{x^2}{2}}$. Observe that $e^{-\frac{x^2}{2}}>0$ for all real values of x.

PINs: -1 and 1.

Concave up on $(-\infty,-1]\cup[1,\infty)$.

Concave down on $[-1,1]$.

Both PINs correspond to IPs: $\left(-1,\dfrac{1}{\sqrt{2e\pi}}\right)$ and $\left(1,\dfrac{1}{\sqrt{2e\pi}}\right)$.



SECTION 7.4: INTEGRATION and LOG / EXP. FUNCTIONS

1)

a) $\frac{1}{2}\ln\left|2x-3\right|+C$, or $\ln\sqrt{\left|2x-3\right|}+C$. (The Power Rule for Logarithms can be used to reexpress some of the other expressions in these answers.)

b) $\frac{1}{7}e^{7x}-\frac{1}{7}e^{-7x}+C$, or $\frac{1}{7}\left(e^{7x}-e^{-7x}\right)+C$, or $\frac{1}{7}\left(e^{7x}-\frac{1}{e^{7x}}\right)+C$, or $\frac{e^{14x}-1}{7e^{7x}}+C$

c) $-\frac{1}{3}\ln\left|\cos(3x)\right|+C$, or $C-\frac{1}{3}\ln\left|\cos(3x)\right|$, or $\frac{1}{3}\ln\left|\sec(3x)\right|+C$

d) $5\ln\left|\sin\left(\frac{x}{5}\right)\right|+C$

e) $\frac{5}{8}\ln\left(4x^2+3\right)+C$. Note: $\frac{5}{8}\ln\left|4x^2+3\right|+C=\frac{5}{8}\ln\left(4x^2+3\right)+C$, because $4x^2+3>0$ for all real values of x.

f) $\frac{1}{15}e^{5x^3}+C$

g) $3\ln\left|\sec\left(\theta^2+e\right)+\tan\left(\theta^2+e\right)\right|+C$

h) $\frac{1}{2}\ln\left(x^2-4x+5\right)+C$. Note: $\frac{1}{2}\ln\left|x^2-4x+5\right|+C=\frac{1}{2}\ln\left(x^2-4x+5\right)+C$, because $x^2-4x+5=\left(x^2-4x+4\right)+1=\left(x-2\right)^2+1>0,\ \forall x\in\mathbb{R}$.

i) $\frac{t^2}{2}+6t+9\ln\left|t\right|+C$, or $\frac{t^2+12t+18\ln\left|t\right|}{2}+C$, or $\frac{t^2+12t+\ln\left(t^{18}\right)}{2}+C$.
Hint: Expand the numerator by performing the indicated square.

j) $-\cot x+8\ln\left|\csc x-\cot x\right|+16x+C$, or $C-\cot x+8\ln\left|\csc x-\cot x\right|+16x$, or $C-\cot x-8\ln\left|\csc x+\cot x\right|+16x$

k) $-\cos\left(\ln x\right)+C$, or $C-\cos\left(\ln x\right)$

l) $\frac{2\pi e^{\sqrt{x}}}{7}+C$

m) $\ln\left|x+\tan x\right|+C$

n) $e^x + 2x - e^{-x} + C$, or $e^x + 2x - \dfrac{1}{e^x} + C$, or $\dfrac{e^{2x} + 2xe^x - 1}{e^x} + C$.

Hint: Expand the numerator by performing the indicated square.

o) $-\dfrac{1}{e^x - e^{-x}} + C$, or $C - \dfrac{e^x}{e^{2x} - 1}$, or $\dfrac{e^x}{1 - e^{2x}} + C$

p) $\ln\left(e^x + 1\right) + C$. Note: $\ln\left| e^x + 1 \right| + C = \ln\left(e^x + 1\right) + C$, because $e^x + 1 > 0$ for all real values of x.

q) $\dfrac{1}{3}\ln\left| \sec\left(3e^x - e\right) + \tan\left(3e^x - e\right) \right| + C$

r) $e^{\sin x} + C$. Hint: Use a Reciprocal Identity.

s) $\ln\left| \csc\theta - \cot\theta \right| + \cos\theta + C$, or $-\ln\left| \csc\theta + \cot\theta \right| + \cos\theta + C$.

Hint: Use a Pythagorean ID.

t) $\dfrac{1}{2}\ln\left| 1 + \csc\left(e^{-2x}\right) \right| + C$, or $\dfrac{1}{2}\ln\left| 1 + \csc\left(\dfrac{1}{e^{2x}}\right) \right| + C$

2) a) 0.697, b) $\ln 2 \approx 0.693$

3) (Left to the reader.)

4) Hint: $-\ln\left| \csc x + \cot x \right| = \ln\left(\left| \csc x + \cot x \right|^{-1} \right) = \ln\left(\dfrac{1}{\left| \csc x + \cot x \right|} \cdot \dfrac{\left| \csc x - \cot x \right|}{\left| \csc x - \cot x \right|} \right)$.

5) $s\left(t\right) = 2e^{2t} - \dfrac{3}{2}e^{-2t} + \dfrac{7}{2}$ in feet, or $s\left(t\right) = \dfrac{4e^{4t} + 7e^{2t} - 3}{2e^{2t}}$ in feet.

6) $\dfrac{\pi\left(e - 1\right)}{e}$ m^3. This is about 1.986 m^3.

Hint: Setup is: $\int_0^1 2\pi x e^{-x^2}\, dx$ from the Cylinder / Cylindrical Shell Method (6.3).

Note 1: Observe that $e^{-x^2} = \dfrac{1}{e^{x^2}} > 0$ for all real values of x.

Note 2: $\pi\left(1 - e^{-1}\right) = \dfrac{\pi\left(e - 1\right)}{e}$.

7) $\dfrac{\pi}{2}\ln 2$ m^3. Observe that both $-\pi\ln\left(\dfrac{\sqrt{2}}{2}\right)$ m^3 and $\pi\ln\left(\sqrt{2}\right)$ m^3 are equivalent to this. This is about 1.089 m^3.

SECTION 7.5: BEYOND *e* – NONNATURAL BASES

1)

a) $ex^{e-1} + e^x + 2^x \ln 2 - 2^{3x^4+x}(\ln 2)(12x^3+1)$

b) $\frac{1}{x} + \frac{1}{x\ln 2}$, or $\frac{1}{x} + \frac{1}{(\ln 2)x}$, or $\frac{\ln 2 + 1}{x \ln 2}$

c) $\frac{4(7x^6 - 12x^2)}{(x^7 - 4x^3 + 2)\ln 10}$, or $\frac{4x^2(7x^4 - 12)}{(x^7 - 4x^3 + 2)\ln 10}$

d) $x^{\pi-1}\pi^{x+1} + x^\pi \pi^x \ln \pi$, or $x^{\pi-1}\pi^x(\pi + x\ln\pi)$.

e) $5 \cdot 3^{\sec(5t)}(\ln 3)\sec(5t)\tan(5t)$, or $(\ln 243)3^{\sec(5t)}\sec(5t)\tan(5t)$

f) $\frac{1}{(\ln 6)r\ln r}$

g) $\left[2x\ln(x+3) + \frac{x^2}{x+3}\right](x+3)^{x^2}$, which can simplify to

$x\left[2(x+3)\ln(x+3) + x\right](x+3)^{x^2-1}$.

h) $\left[(\sec^2 x)\ln x + \frac{\tan x}{x}\right]x^{\tan x}$, which can simplify to

$\left[x(\sec^2 x)\ln x + \tan x\right]x^{\tan x - 1}$.

2) (Your answer should be equivalent to the one for Exercise 1h.)

3)

a) $\frac{x^{e+1}}{e+1} + e^e x + e^x + \frac{2^x}{\ln 2} - \frac{9\pi^x}{\ln \pi} + C$

b) $\frac{7^{5x+3}}{5\ln 7} + C$, or $\frac{7^{5x+3}}{\ln 7^5} + C = \frac{7^{5x+3}}{\ln 16{,}807} + C$

c) $(\ln 10)\ln|\log x| + C$, or $(\ln 10)\ln|\ln x| + C$. (These are equivalent by the Change-of-Base Formula and the Quotient Rule for Logarithms. Remember that C can "absorb" constant terms.)

4) $\frac{\ln 11 - \ln 2}{\ln 10}$, or $\frac{\ln\left(\frac{11}{2}\right)}{\ln 10}$, or $\log\left(\frac{11}{2}\right)$ by the Change-of-Base Formula.

CHAPTER 8: INVERSE TRIGONOMETRIC and HYPERBOLIC FUNCTIONS

SECTION 8.1: INVERSE TRIGONOMETRIC FUNCTIONS

1) Evaluate the following. If an expression is not defined as a real number, write "undefined."

   a) $\sin^{-1}\left(\frac{1}{2}\right)$, also known as $\arcsin\left(\frac{1}{2}\right)$

   b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, also known as $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

   c) $\sin^{-1}\left(\frac{\pi}{2}\right)$, also known as $\arcsin\left(\frac{\pi}{2}\right)$

   d) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, also known as $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

   e) $\tan^{-1}(-1)$, also known as $\arctan(-1)$

   f) $\sin^{-1}\left[\sin\left(\frac{\pi}{5}\right)\right]$, also known as $\arcsin\left[\sin\left(\frac{\pi}{5}\right)\right]$

   g) $\cos^{-1}\left[\cos\left(\frac{7\pi}{6}\right)\right]$, also known as $\arccos\left[\cos\left(\frac{7\pi}{6}\right)\right]$

2) Rewrite the following as algebraic expressions in x. Assume $x > 0$.

   a) $\tan\left[\sin^{-1}\left(\frac{x}{5}\right)\right]$, also known as $\tan\left[\arcsin\left(\frac{x}{5}\right)\right]$

   b) $\cos\left[\tan^{-1} x\right]$, also known as $\cos\left[\arctan x\right]$

KNOW THE GRAPHS, DOMAINS, AND RANGES OF THE THREE KEY INVERSE TRIGONOMETRIC FUNCTIONS!

SECTION 8.2: CALCULUS and INVERSE TRIGONOMETRIC FUNCTIONS

1) Find the following derivatives. Simplify where appropriate.
You do not have to simplify radicals or rationalize denominators.

   a) Let $f(x) = \arctan\left(\sqrt{x}\right)$, also written as $\tan^{-1}\left(\sqrt{x}\right)$. Find $f'(x)$.

   b) Find $\frac{d}{dx}\left[x^2 \arcsin(3x)\right]$, also written as $\frac{d}{dx}\left[x^2 \sin^{-1}(3x)\right]$.

   c) Find $D_x\left[\arccos(\ln x)\right]$, also written as $D_x\left[\cos^{-1}(\ln x)\right]$.

   d) Let $g(t) = \left[e^t + \operatorname{arcsec}\left(t^4\right)\right]^5$, also written as $g(t) = \left[e^t + \sec^{-1}\left(t^4\right)\right]^5$.
   Find $g'(t)$. Assume that the range of the arcsec function is given by $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, as we do in the notes. Bear in mind that the range is defined differently in other sources.

   e) Let $y = \arcsin(\arctan x)$, also written as $\sin^{-1}\left(\tan^{-1} x\right)$. Find $\frac{dy}{dx}$.

2) We will find $D_x\left[\arcsin(\sin x)\right]$, also written as $D_x\left[\sin^{-1}(\sin x)\right]$, in two different ways. Assume $-\frac{\pi}{2} < x < \frac{\pi}{2}$; the importance of this assumption is part of the point of this problem.

   a) Simplify first before finding the derivative.

   b) Do not simplify first before finding the derivative.

3) Yes or No: Is $D_x\left[(\sin x)^{-1}\right]$ equivalent to $D_x\left(\sin^{-1} x\right)$?

4) Evaluate the following indefinite integrals. You may use C, D, etc. as representing arbitrary constants.

• Assume that the range of the inverse secant (arcsecant) function is given by $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, as we do in the notes. Bear in mind that the range is defined differently in other sources.

a) $\displaystyle\int \frac{5}{\sqrt{1-t^2}}\, dt$

b) $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\, dx$

c) $\displaystyle\int \frac{x}{\sqrt{16-x^2}}\, dx$

d) $\displaystyle\int \frac{dx}{25+x^2}$

e) $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\, dx$

f) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx$

g) $\displaystyle\int \frac{3x}{100+x^4}\, dx$

h) $\displaystyle\int \frac{\arctan x}{1+x^2}\, dx$

i) $\displaystyle\int \frac{1}{x\sqrt{x^8-9}}\, dx$. Hint: After deciding on a u-sub, multiply the numerator and the denominator by the same expression.

5) Evaluate the definite integral $\displaystyle\int_0^3 \frac{1}{x^2+9}\, dx$. Observe that we can find the **exact** value of this integral; in Chapter 5, we would have numerically approximated it using Riemann sums, the Trapezoidal Rule, or Simpson's Rule.

SECTION 8.3: HYPERBOLIC FUNCTIONS

1) Evaluate $\sinh(1)$, $\cosh(1)$, and $\tanh(1)$. Round off to four significant digits.

2) Prove that $D_x(\sinh x) = \cosh x$.

3) Prove that $\cosh^2 x - \sinh^2 x = 1$.

4) Find the following derivatives. Simplify where appropriate.

   a) $D_x\left[\sinh(3x)\right]$

   b) $D_x\left[\cosh(3x)\right]$

   c) $D_x\left[4\tanh\left(e^x\right) - 1\right]$

   d) $D_x\left[x\ln(\operatorname{sech} x)\right]$

   e) $D_x\left[3x^2 + 2^{\operatorname{csch} x}\right]$

   f) $\dfrac{d}{dt}\left(\left[\coth(\sec t)\right]^4\right)$

   g) $D_x\left(\dfrac{\cosh x}{\arctan x}\right)$

5) Evaluate the following indefinite integrals.

   a) $\int \cosh(3x)\,dx$. Try using Guess-and-Check here.

   b) $\int \sinh(3x)\,dx$. Try using Guess-and-Check here.

   c) $\displaystyle\int \frac{7x}{\cosh^2\left(4x^2 - 1\right)}\,dx$

   d) $\displaystyle\int \frac{\operatorname{sech}\left(\sqrt{x}\right)\tanh\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

   e) $\int e^t \coth\left(e^t\right)\operatorname{csch}^2\left(e^t\right)\,dt$

CHAPTER 8: INVERSE TRIGONOMETRIC and HYPERBOLIC FUNCTIONS

SECTION 8.1: INVERSE TRIGONOMETRIC FUNCTIONS

1)

a) $\dfrac{\pi}{6}$

b) $-\dfrac{\pi}{3}$

c) undefined

d) $\dfrac{3\pi}{4}$

e) $-\dfrac{\pi}{4}$

f) $\dfrac{\pi}{5}$

g) $\dfrac{5\pi}{6}$

2)

a) $\dfrac{x}{\sqrt{25-x^2}}$, or $\dfrac{x\sqrt{25-x^2}}{25-x^2}$

b) $\dfrac{1}{\sqrt{x^2+1}}$, or $\dfrac{\sqrt{x^2+1}}{x^2+1}$

SECTION 8.2: CALCULUS and INVERSE TRIGONOMETRIC FUNCTIONS

1)

a) $\dfrac{1}{2\sqrt{x}\left(1+x\right)}$, or $\dfrac{\sqrt{x}}{2x\left(1+x\right)}$

b) $2x\arcsin\left(3x\right)+\dfrac{3x^2}{\sqrt{1-9x^2}}$, or $\dfrac{\sqrt{1-9x^2}\left[2x\sqrt{1-9x^2}\arcsin\left(3x\right)+3x^2\right]}{1-9x^2}$

c) $-\dfrac{1}{x\sqrt{1-\left(\ln x\right)^2}}$

d) $5\left[e^t+\operatorname{arcsec}\left(t^4\right)\right]^4\left[e^t+\dfrac{4}{t\left(\sqrt{t^8-1}\right)}\right]$, or $\dfrac{5\left[e^t+\operatorname{arcsec}\left(t^4\right)\right]^4\left(te^t\sqrt{t^8-1}+4\right)}{t\left(t^8-1\right)}$

e) $\dfrac{1}{\left(1+x^2\right)\sqrt{1-\left(\arctan x\right)^2}}$

2) a) and b) 1.

In a), use inverse properties. Observe that $\arcsin\left(\sin x\right)=x$ if $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$.

Note: There are other real values of x for which $\arcsin\left(\sin x\right)\neq x$;

for example, consider $x=\dfrac{5\pi}{6}$.

In b), observe that $\sqrt{1-\sin^2 x}=\sqrt{\cos^2 x}=\left|\cos x\right|=\cos x$, because $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$.

3) No. The "-1" superscript is an exponent (indicating reciprocal or multiplicative inverse) in $(\sin x)^{-1}$, but it indicates an inverse function in $\sin^{-1} x$.

$D_x\left[(\sin x)^{-1}\right] = D_x\left[\dfrac{1}{\sin x}\right] = D_x\left[\csc x\right] = -\csc x \cot x$.

This is not equivalent to $D_x\left(\sin^{-1} x\right)$, which is $\dfrac{1}{\sqrt{1-x^2}}$.

4)

a) $5 \arcsin t + C$, or $5 \sin^{-1} t + C$

b) $\arcsin\left(\dfrac{x}{4}\right) + C$, or $\sin^{-1}\left(\dfrac{x}{4}\right) + C$

c) $-\sqrt{16-x^2} + C$, or $C - \sqrt{16-x^2}$. Hint: Use a u-sub instead of the techniques of this chapter.

d) $\dfrac{1}{5}\arctan\left(\dfrac{x}{5}\right) + C$, or $\dfrac{1}{5}\tan^{-1}\left(\dfrac{x}{5}\right) + C$

e) $\dfrac{1}{2}\operatorname{arcsec}\left(\dfrac{x}{2}\right) + C$, or $\dfrac{1}{2}\sec^{-1}\left(\dfrac{x}{2}\right) + C$

f) $\arcsin\left(e^x\right) + C$, or $\sin^{-1}\left(e^x\right) + C$

g) $\dfrac{3}{20}\arctan\left(\dfrac{x^2}{10}\right) + C$, or $\dfrac{3}{20}\tan^{-1}\left(\dfrac{x^2}{10}\right) + C$

h) $\dfrac{(\arctan x)^2}{2} + C$, or $\dfrac{\left(\tan^{-1} x\right)^2}{2} + C$

i) $\dfrac{1}{12}\operatorname{arcsec}\left(\dfrac{x^4}{3}\right) + C$, or $\dfrac{1}{12}\sec^{-1}\left(\dfrac{x^4}{3}\right) + C$.

Hint: If $u = x^4$, then $du = 4x^3\,dx$. Multiply the numerator and the denominator by $4x^3$.

5) $\dfrac{\pi}{12}$

SECTION 8.3: HYPERBOLIC FUNCTIONS

1) $\sinh(1) \approx 1.175$, $\cosh(1) \approx 1.543$, $\tanh(1) \approx 0.7616$

2) Prove that $D_x\left(\dfrac{e^x - e^{-x}}{2}\right) = \dfrac{e^x + e^{-x}}{2}$.

3) Prove that $\left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = 1$.

4)

a) $3\cosh(3x)$

b) $3\sinh(3x)$

c) $4e^x \operatorname{sech}^2(e^x)$

d) $\ln(\operatorname{sech} x) - x\tanh x$

e) $6x - 2^{\operatorname{csch} x}(\ln 2)\operatorname{csch} x \coth x$

f) $-4\sec t \tan t \coth^3(\sec t)\operatorname{csch}^2(\sec t)$

g) $\dfrac{(1+x^2)\arctan x \sinh x - \cosh x}{(1+x^2)(\arctan x)^2}$. Hint: $\dfrac{\arctan x \sinh x - \dfrac{\cosh x}{1+x^2}}{(\arctan x)^2}$ simplifies to this.

5)

a) $\dfrac{1}{3}\sinh(3x) + C$

b) $\dfrac{1}{3}\cosh(3x) + C$

c) $\dfrac{7}{8}\tanh(4x^2 - 1) + C$

d) $-2\operatorname{sech}(\sqrt{x}) + C$

e) $-\dfrac{1}{2}\coth^2(e^t) + C$, or $C - \dfrac{\coth^2(e^t)}{2}$, or $C - \dfrac{\operatorname{csch}^2(e^t)}{2}$